AN ITERATIVE ALGORITHM FOR FINDING THE MINIMUM SAMPLING FREQUENCY FOR TWO BANDPASS SIGNALS

Yuan-Pei Lin, Yi-De Liu
Dept. Elect. and Control Engr.,
National Chiao Tung Univ.,
Hsinchu, Taiwan

See-May Phoong
Dept. of EE & Grad. Inst. of Comm Engr.,
National Taiwan Univ.,
Taipei, Taiwan

ABSTRACT

In this paper, we propose an efficient algorithm for finding the minimum sampling frequency for a signal that consists of two bandpass signals. This has important application in software radio where it is desirable to downconvert multiple bandpass signals simultaneously. We will derive a new set of conditions for alias-free sampling that can be checked with few computations. The minimum sampling frequency can be found by iteratively increasing the sampling frequency to meet the alias-free conditions. We will demonstrate that the proposed method has a much lower complexity than previously reported algorithms.

1. INTRODUCTION

Bandpass sampling has important applications in downcoverting radio frequency (RF) signals. In the application of software defined radio systems, it is desirable to downconvert multiple RF signals simultaneously to save cost [1, 2]. The signal to be sampled may consist of more than one bandpass signal. An example of spectrum that contains two bandpass signals (four passbands) is shown in Fig. 1. Sampling theorem for a bandpass signal (two passbands) is well-known [3]. The minimum frequency for alias-free sampling can be found in a closed form [4]. The minimum sampling frequency is usually significantly lower than the carrier frequency of the bandpass signal.

For signals with more than two passbands, the minimum sampling frequency can not be found in a closed from due to the nonlinear nature of spectrum folding in the process of sampling. Sampling for multi-band signals is extended in [2]. Conditions for alias-free sampling of multi-band signals are derived [2]. A systematic algorithm for finding valid sampling frequencies is developed in [5]. In [6][7], the complexity for finding valid sampling frequency is considerably reduced by imposing constraints on the ordering of the bands in the folded spectrum. These results may not yield the minimum frequency for alias-free sampling due to the ordering constraints. An efficient algorithm for finding valid sampling frequency range is proposed in [8]. By exhausting all possible orderings of the bands in the folded spectrum and categorizing all possible cases, the computational complexity can be reduced. An algorithm for finding the minimum sampling frequency is developed in [9] by finding the intersection of valid sampling frequencies for every two signal bands.

In this paper we propose an efficient algorithm for finding the minimum sampling frequency for a signal consisting of two bandpass signals. We will first derive a set of conditions for alias-free sampling that can be checked with few computations. We will show how the sampling frequency can be increased with minimum increment to satisfy each of these conditions. By iteratively meeting the conditions for alias-free sampling, an algorithm for finding the minimum sampling frequency can be developed. There is no need to consider ordering of the signal bands in the folded spectrum. We will see that the algorithm based on the conditions derived in this paper requires fewer computations when compared to previously reported methods.

The rest of the paper is organized as follows. The conditions for alias-free sampling are derived in Sec. 2. The algorithm for finding the minimum sampling frequency is given in Sec. 3. Simulation examples are presented in Sec. 4 and a conclusion is given in section 5.

2. CONDITIONS FOR ALIASFREE SAMPLING

Conditions for alias-free sampling can be stated in various ways in terms of the band edges and bandwidths of the member bandpass signals. The conditions that are employed affect the complexity of ensuing algorithms. In this section, we derive a new set of conditions for aliasfree sampling that will lead to an efficient algorithm in the next section.

Suppose we are to sample a signal $X(f)$ that consists of two bandpass signals $X_1(f)$ and $X_2(f)$ as shown in Fig. 1. Assume $X_i(f) \neq 0$ for $f_{li} < |f| < f_{h_i}$, $i = 1, 2$, where f_{ℓ_i} and f_{h_i} are band edges, and $W_i = f_{h_i} - f_{\ell_i}$ are one-

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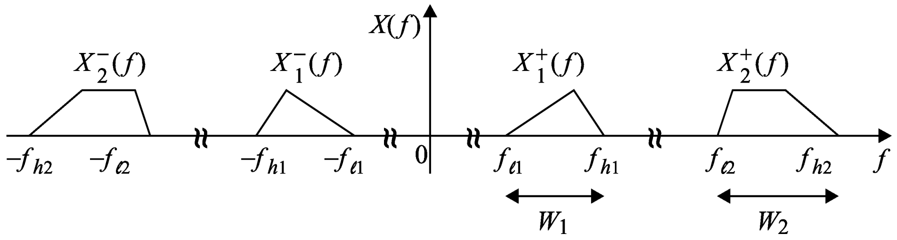

Figure 1: An example of spectrum that consists of two bandpass signals.

sided bandwidths as indicated in the figure. Let $X_i^+(f)$, and $X_i^-(f)$ denote respectively the positive frequency part and negative frequency part of $X_i(f)$. There are four signal bands, including $X_1^+(f)$, $X_1^-(f)$, $X_2^+(f)$, and $X_2^-(f)$. Since the replicas of any two bands may overlap and result in aliasing after sampling, there are a total of $C_2^4 = 6$ cases. Note that $X_1^+(f)$ and $X_1^-(f)$ are symmetric with respect to 0, and so are $X_2^+(f)$ and $X_2^-(f)$. If $X_1^+(f)$ and $X_2^+(f)$ are not aliasing after sampling, then $X_1^-(f)$ and $X_2^-(f)$ will not be aliasing by symmetry. Similarly, if $X_1^-(f)$ and $X_2^+(f)$ are not aliasing after sampling, then $X_1^+(f)$ and $X_2^-(f)$ will not be aliasing. Thus, we need to consider only 4 cases:

(a) $\{X_1^+(f), X_1^-(f)\}$

(b) $\{X_2^+(f), X_2^-(f)\}$

(c) $\{X_1^+(f), X_2^+(f)\}$

(d) $\{X_1^-(f), X_2^+(f)\}$.

Case (a). If we consider only the pair $\{X_1^+(f), X_1^-(f)\}$ as shown in Fig. 2(a), this is the same as the case of one bandpass signal. For convenience, we will derive a condition in terms of the band edge f_{h_1} and one-sided bandwidth W_1. Upon sampling with frequency f_s, replicas of $X_1^+(f)$ and $X_1^-(f)$ appear every f_s, resulting in a periodic spectrum; we can simply consider the period $[0, f_s)$. Since $X_1^+(f)$ and $X_1^-(f)$ are symmetric with respect zero, the replicas of $X_1^+(f)$ and $X_1^-(f)$ are symmetric with respect to $\frac{f_s}{2}$ in the interval $[0, f_s)$ (Fig. 2(b)). Observe that if 0 or f_s is not contained inside the band of replicas of $X_1^+(f)$ and $X_1^-(f)$ there will not be aliasing. One necessary and sufficient condition for alias-free sampling is thus $f_{h_1} \pmod{\frac{f_s}{2}} = 0$, or $f_{h_1} \pmod{\frac{f_s}{2}} \geq W$. Equivalently, we have

$$\begin{array}{ll} & 2f_{h_1} \pmod{f_s} = 0 \\ \text{or} & 2f_{h_1} \pmod{f_s} \geq 2W_1 \end{array} \quad (1)$$

Case (b). Similar to case (a), if we consider the pair

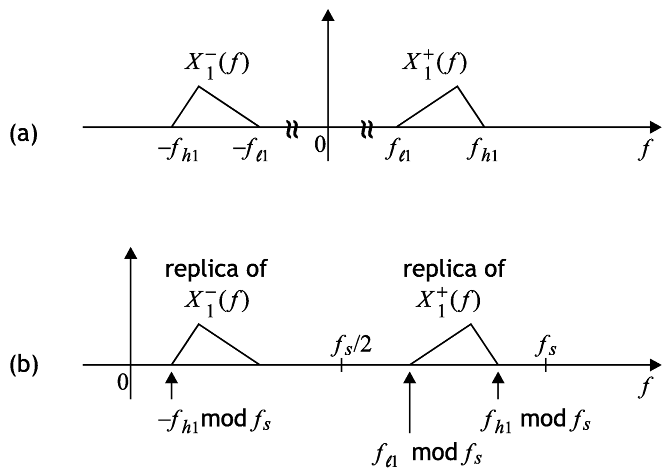

Figure 2: (a) The spectrum of $X_1^+(f)$ and $X_1^-(f)$. (b) An example of the folded spectrum for the interval $[0, f_s)$.

$\{X_2^+(f), X_2^-(f)\}$, there will be no aliasing if and only if

$$\begin{array}{ll} & 2f_{h_2} \pmod{f_s} = 0, \\ \text{or} & 2f_{h_2} \pmod{f_s} \geq 2W_2 \end{array} \quad (2)$$

Case (c). Consider Fig. 3(a) where we have shown only the pair $\{X_1^+(f), X_2^+(f)\}$. First observe that there is no aliasing due to this pair if and only if there is no aliasing when we sample a shifted version of the pair $\{X_1^+(f + f_0), X_2^+(f + f_0)\}$ where f_0 is the shift. For convenience we will consider the condition for alias-free sampling of the pair with a shift. Suppose we choose f_0 as the midpoint of f_{ℓ_1} and f_{h_2}, i.e.,

$$f_0 = (f_{\ell_1} + f_{h_2})/2.$$

Then the shifted pair is as shown in Fig. 3(b), where

$$a = \frac{f_{h_2} - f_{\ell_1}}{2},$$

$$b = f_{\ell_2} - (f_{\ell_1} + f_{h_2})/2,$$

$$c = f_{h_1} - (f_{\ell_1} + f_{h_2})/2.$$

If we consider the folded spectrum in the $[0, f_s)$ interval, the band edges $a \pmod{f_s}$ and $-a \pmod{f_s}$ are equal-distanced from $f_s/2$. We now discuss two possible scenarios (i) $a \pmod{f_s} \geq -a \pmod{f_s}$ and (ii) $a \pmod{f_s} < -a \pmod{f_s}$. Examples of these two possible cases are shown respectively in Fig. 3(c) and (d).

(i) When $a \pmod{f_s} \geq -a \pmod{f_s}$ there will be no aliasing if and only if $-a \pmod{f_s} = a \pmod{f_s}$ or if the interval $(-a \pmod{f_s}, a \pmod{f_s})$ is large enough to accommodate the two replicas. That is,

$$\begin{aligned} & a \pmod{f_s} - (-a \pmod{f_s}) = 0, \\ \text{or} \quad & a \pmod{f_s} - (-a \pmod{f_s}) \geq W_1 + W_2. \end{aligned}$$

The equivalent conditions are

$$\begin{aligned} & 2a \pmod{f_s} = 0, \\ \text{or} \quad & 2a \pmod{f_s} \geq W_1 + W_2 \end{aligned} \tag{3}$$

(ii) when $a \pmod{f_s} < -a \pmod{f_s}$ as shown in Fig. 3(d), there is some space between the two replicas and the space is of length $(-a \pmod{f_s} - a \pmod{f_s})$. There will be no aliasing if and only if the remaining part of the $[0, f_s)$ interval is large enough to take in the two replicas. That is,

$$f_s - (-a \pmod{f_s} - a \pmod{f_s}) \geq W_1 + W_2.$$

Or equivalently

$$2a \pmod{f_s} \geq W_1 + W_2$$

This is the same as the second condition in (3).

Substituting $a = (f_{h_2} - f_{\ell_1})/2$ to (3), we obtain one necessary and sufficient condition for alias-free sampling

$$\begin{aligned} & (f_{h_2} - f_{\ell_1}) \pmod{f_s} = 0, \\ \text{or} \quad & (f_{h_2} - f_{\ell_1}) \pmod{f_s} \geq W_1 + W_2 \end{aligned} \tag{4}$$

Case (d). Similarly, for the pair $\{X_1^-(f), X_2^+(f)\}$, we can use the technique in case (c) to find the following necessary and sufficient condition for alias-free sampling

$$\begin{aligned} & (f_{h_1} + f_{h_2}) \pmod{f_s} = 0, \\ \text{or} \quad & (f_{h_1} + f_{h_2}) \pmod{f_s} \geq W_1 + W_2 \end{aligned} \tag{5}$$

Summarizing, for given a sampling frequency f_s, there will not be aliasing if the following four conditions are satisfied.

1. $2f_{h_1} \pmod{f_s} = 0$ or $2f_{h_1} \pmod{f_s} \geq 2W1$
2. $2f_{h_2} \pmod{f_s} = 0$ or $2f_{h_2} \pmod{f_s} \geq 2W2$
3. $(f_{h_2} - f_{\ell_1}) \pmod{f_s} = 0$ or $(f_{h_2} - f_{\ell_1}) \pmod{f_s} \geq W_1 + W_2$
4. $(f_{h_1} + f_{h_2}) \pmod{f_s} = 0$ or $(f_{h_1} + f_{h_2}) \pmod{f_s} \geq W_1 + W_2$

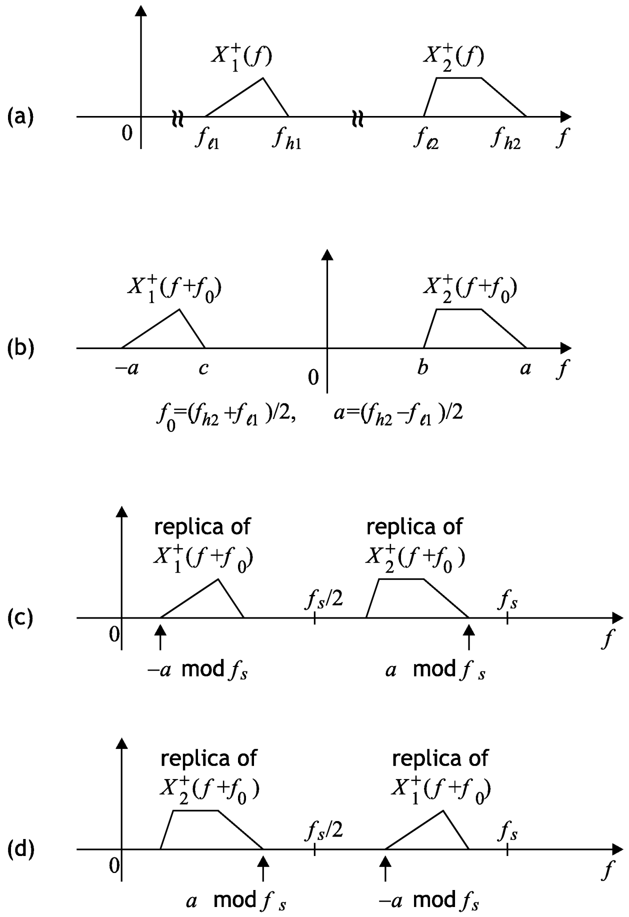

Figure 3: (a) The spectrum of $X_1^+(f)$ and $X_2^+(f)$. (b) The shifted spectrum $X_1^+(f + f_0)$ and $X_2^+(f + f_0)$, where $f_0 = (f_{h_2} + f_{\ell_1})/2$ and $a = (f_{h_2} - f_{\ell_1})/2$. (c) An example of the folded spectrum for the interval $[0, f_s)$ when $a \pmod{f_s} \geq -a \pmod{f_s}$. (d) An example of the folded spectrum for the interval $[0, f_s)$ when $a \pmod{f_s} < -a \pmod{f_s}$.

3. PROPOSED ALGORITHM FOR FINDING THE MINIMUM SAMPLING FREQUENCY

First for each of the four cases in Sec. 2 we derive the minimum increment in sampling frequency such that the corresponding condition for alias-free sampling can be satisfied.

Case (a). Suppose the condition in (1) is not satisfied for a given sampling frequency f_s. Consider the folded spectrum for the interval $[0, f_s)$. We discuss the two cases (i) $0 < f_{h_1} \pmod{f}_s < f_s/2$ and (ii) $f_s/2 < f_{h_1} \pmod{f}_s > f_s$ separately.

(i) $0 < f_{h_1} \pmod{f}_s < f_s/2$: When we gradually increase the sampling frequency the band edge $f_{h_1} \pmod{f_s}$ of replica $X_1^+(f)$ moves towards 0 while the band edge $-f_{h_1} \pmod{f_s}$ of replica $X_1^-(f)$ moves towards f_s. When the sampling frequency is increased such that $f_{h_1} \pmod{f_s}$ decreases to 0, then the condition in (1) becomes satisfied.

(ii) $f_s/2 < f_{h_1} \pmod{f_s} < f_s$: Similarly the condition in (1) becomes satisfied when $f_{h_1} \pmod{f_s}$ decreases to $f_s/2$.

Therefore we can conclude that the alias-free condition (1) can be satisfied by increasing the sampling frequency such that f_{h_1} becomes an integer multiple of $f_s/2$. The smallest new sampling $f_{s,new}$ for this to happen can be computed as follows. Let

$$f_{h_1} = n_{h_1} f_s/2 + r_{h_1},$$

where $r_{h_1} = f_{h_1} \pmod{f_s/2}$ and $n_{h_1} = \lfloor f_{h_1}/(f_s/2) \rfloor$ with the notation $\lfloor x \rfloor$ denoting the largest integer smaller than or equal to x. Then we have $f_{h_1} = n_{h_1} f_{s,new}/2$, or equivalently

$$f_{s,new} = \frac{2f_{h_1}}{n_{h_1}} = \frac{2f_{h_1}}{\lfloor f_{h_1}/(f_s/2) \rfloor} = \frac{2f_{h_1}}{\lfloor 2f_{h_1}/f_s \rfloor}, \tag{6}$$

where we have used the fact that n_{h_1} can also be computed using $n_{h_1} = \lfloor 2f_{h_1}/f_s \rfloor$.

Case (b). Similar to case (a), if the condition in (2) is not satisfied, we can increase sampling frequency to

$$f_{s,new} = \frac{2f_{h_2}}{\lfloor 2f_{h_2}/f_s \rfloor}, \tag{7}$$

then (2) will become satisfied.

Case (c). Suppose the condition in (4) is not satisfied. Consider again the shifted spectrum in Fig. 3(b). Using the steps in case (a), we can verify that there will be not aliasing if we increase the sampling frequency so that $a \pmod{f_s}$ to be equal to 0 or $\frac{f_s}{2}$. Moreover the new sampling frequency can be obtained by

$$f_{s,new} = \frac{2a}{\lfloor a/\frac{f_s}{2} \rfloor} = \frac{f_{h_2} - f_{\ell_1}}{\lfloor (f_{h_2} - f_{\ell_1})/f_s \rfloor} \tag{8}$$

Case (d). Like case (c), if the condition in (5) is not satisfied, we can increase the sampling frequency to

$$f_{s,new} = \frac{f_{h_1} + f_{h_2}}{\lfloor (f_{h_1} + f_{h_2})/f_s \rfloor} \tag{9}$$

then (5) will be satisfied.

Using the conditions for alias-free sampling in Sec. 2 and the methods for computing new sampling frequency for each case, we have the following iterative algorithm for finding the minimum sampling frequency. To start off, let $f_s = 2(W_1 + W_2)$, which is the lowest possible sampling frequency for no aliasing.

1. Examine if the condition for case (a) in (1) is satisfied. If it is, go to the next step. If it is not satisfied, compute the new sampling frequency using (6) and go to the next step.

2. If the condition (2) for case (b) is satisfied, go to the next step. If it is not satisfied, compute the new sampling frequency using (7) and go to step 1.

3. If the condition (4) for case (c) is satisfied, go to the next step. If it is not, compute the new sampling frequency using (8) and go to step 1.

4. If the condition (5) for case (d) is not satisfied, compute the new sampling frequency using (9) and go to step 1. If it is satisfied then we have found the minimum sampling frequency.

Usually not all four steps are performed in one iteration.

Remark on complexity. The main computations are in the inspection of conditions in (1), (2), (4) and (5), and the computation of new sampling frequency in (6)-(9). Few computations are required for these equations as we can borrow results from earlier evaluations. For example in step 1 we compute $2f_{h_1} \pmod{f_s}$ in (1). In the process we can also obtain the integer n_{h_1} which is used in computing the new sampling frequency (6). Similar conclusions can be drawn for steps 2-4. In step 3, we need to evaluate $f_{h_2} - f_{\ell_1} \pmod{f_s}$ which can be written as

$$\begin{aligned} & f_{h_2} - f_{\ell_1} \pmod{f_s} \\ = & \underbrace{(f_{h_2} \pmod{f_s} - f_{\ell_1} \pmod{f_s})}_{\text{call this } x} \pmod{f_s} \\ = & \begin{cases} x & , \quad x \geq 0, \\ x + f_s & , \quad \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

With conditions in step 1 already satisfied, we can obtain $f_{\ell_1} \pmod{f_s}$ using

$$f_{\ell_1} \pmod{f_s} = f_{h_1} \pmod{f_s} - W_1.$$

Both $f_{h_1} \pmod{f_s}$ and $f_{h_2} \pmod{f_s}$ can be obtained from steps 1 and 2. The evaluation requires at most 3 additions. Similarly we can verify that in step 4 the evaluation of $f_{h_1} + f_{h_2} \pmod{f_s}$ requires at most 2 additions.

4. SIMULATIONS AND COMPARISONS

In this section, we apply the proposed algorithm to wireless applications. The bandpass signals considered in the simulations are GSM 900 (935-960 MHz, one-sided bandwidth 25 MHz), GSM 1800 (1805-1880 MHz, one-sided bandwidth 75 MHz) [11], DAB Eureka-147 L-Band (1472.286-1473.822 MHz, one-sided bandwidth 1536 KHz) [12], IEEE

Case	Method in [8]		Method in [9]		Proposed Method	
	Additions	Multiplications	Additions	Multiplications	Additions	Multiplications
GSM900, GSM1800	128	96	93	62	14	28
DAB, 802.11g	320	240	462	308	35	88
GSM900, WCDMA	160	120	282	188	17	29
DAB, WCDMA	608	456	1335	890	64	169

Table 1: Complexity for finding the minimum sampling frequency for 4 applications.

802.11g (2412-2432 MHz, one-sided bandwidth 20 MHz) [13], and WCDMA (2119-2124 MHz, one-sided bandwidth 5 MHz).

Table 1 lists the complexity in finding the minimum sampling frequency for 4 different combinations of bandpass signals. For comparison, we have also shown the complexity of the methods in [8][9]. The simulation result demonstrates that the proposed method can reduce the number of additions and multiplications significantly. The required numbers of additions and multiplications are reduced respectively by around 89% and 60-75% compared with the other two methods. It is much more efficient for finding the minimum bandpass sampling frequency.

5. CONCLUSIONS

We have proposed an efficient algorithm for finding the minimum sampling frequency for signals that contain two bandpass signals (four passbands). We have derived a set of necessary and sufficient conditions on the sampling frequency for alias-free sampling. The conditions can be checked with few computations. Our proposed algorithm finds the minimum sampling frequency by iteratively increasing the sampling frequency to meet the alias-free conditions. The complexity for finding the minimum sampling frequency is much lower than existing methods. As there is no need to consider ordering of the signal bands in the folded spectrum, it is easier to extend the proposed method to sampling for multi-bandpass signals.

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