PERIODICALLY NONUNIFORM SAMPLING OF A NEW CLASS OF BANDPASS SIGNALS

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Abstract. It is known that a continuous time signal $x(t)$ with Fourier transform $X(\nu)$ band-limited to $|\nu| < \Theta/2$ can be reconstructed from its samples $x(T_0n)$ with $T_0 = 2\pi/\Theta$. In the case that $X(\nu)$ consists of two bands and is band-limited to $\nu_0 < |\nu| < \nu_0+\Theta/2$, successful reconstruction of $x(t)$ from $x(T_0n)$ requires that these two bands be located properly. When the two bands are not located properly, Kohlenberg showed that we can use a Periodically Nonuniform Sampling (PNS) scheme to recover $x(t)$. In this paper, we show that PNS scheme can be generalized and applied to a wider class. Further generalizations will be made to two-dimensional case and discrete-time case.

1. INTRODUCTION

It is well-known that successful reconstruction of a continuous-time bandpass signal $x(t)$ (Fig. 1) from samples $x(nT_0)$, where $T_0 = 2\pi/\Theta$, depends on the relative positions of these two bands [1]. A necessary and sufficient condition is that, the band edge ν_0 must be an integer multiple of $\Theta/2$. It can be shown that a much wider class of signals with total bandwidth Θ can be recovered from samples at nT_0. To be more specific, define the support of $X(\nu)$ (denoted by $Supp\{X(\nu)\}$) to be the set of frequencies for which $X(\nu) \neq 0$. Then $x(t)$ can be obtained from $x(nT_0)$ if and only if no two frequencies in $Supp\{X(\nu)\}$ overlap under modulo Θ operation [2]. Such signals are called aliasfree(T_0) and their supports are referred to as aliasfree(T_0) zones.

When the two bands of $X(\nu)$ (Fig. 1) are not properly located, Kohlenberg [3] proposed a periodically nonuniform sampling approach to recover $x(t)$(Fig. 2 with $L = 2$). In this scheme two sets of samples, $x(nT)$ and $x(nT + d_1)$, where $T = 2T_0$, as shown in Fig. 3, are used. The average sampling rate is still Θ. Then $x(t)$ can be reconstructed by properly choosing d_1 and the synthesis filters $f_0(t)$ and $f_1(t)$ [3]. This is called periodically nonuniform sampling of second order (PNS(2)), [4], for there are two sets of uniform samples involved. Recently, general Lth order periodically nonuniform sampling (PNS(L)) for such two bands signals has been considered in [5].

In discrete time case, sampling is replaced by decimation. PNS(L) sampling retains L sets of samples, $x(Mn + d_0)$, $x(Mn + d_1)$, $\cdots$, $x(Mn + d_{L-1})$ or the d_0th, d_1th, $\ldots$, d_{L-1}th polyphase components [6]. In [7], PNS(L) sampling and reconstruction has been considered for a subclass of L-band signals. The subclass addressed therein are those whose frequency supports are the union of L bands, each band with bandwidth $2\pi/M$ and band edges at integer multiples of $2\pi/M$. Such a L-band sequence $x(n)$ can be reconstructed from its first L polyphase components, i.e., $x(Mn)$, $x(Mn+1)$, $\cdots$, $x(Mn+L-1)$ [7].

In this paper, we will generalize the results in [3] and [7] to a wider class of signals in terms of frequency supports. We will show that from PNS(L) samples we can reconstruct signals in the class $U(T, L)$, which is the collection of signals whose supports are the union of L non overlapping aliasfree(T) sets [8].[‡] The discrete-time version of these will also be addressed. We will see that 1D discrete-time $U(M, L)$ sequences can always be reconstructed from their first L polyphase components. However, in 2D discrete-time case *only a subclass of* $U(\mathbf{M}, L)$ *signals allows reconstruction from* L *polyphase components.*

Notations

1. The support of $X(\nu)$ (denoted by $Supp\{X(\nu)\}$) is defined as the set of frequencies for which $X(\nu) \neq 0$.
2. A set S is called an aliasfree(T) zone if no two frequencies in S overlap under modulo $2\pi/T$ operation. When $Supp\{X(\nu)\}$ is an aliasfree(T) zone, $x(t)$ is called aliasfree(T).
3. The notation $U(T, L)$ represents the collection of signals whose frequency supports are the union of L non overlapping aliasfree(T) sets.

2. PERIODICALLY NONUNIFORM SAMPLING OF LTH ORDER—CONTINUOUS TIME

In this section, we consider periodically nonuniform sampling of Lth order (PNS(L)) for the class $U(T, L)$. In PNS(L) sampling of $x(t)$, there are L sets of samples, $x(nT)$, $x(nT + d_1)$, $\cdots$, $x(nT + d_{L-1})$. Referring to Fig. 2, the sampling rate is $\sigma = 2\pi/T$ in each channel

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‡ Throughout this paper, we will assume that aliasfree(T) sets contain only finitely many intervals.

and the average sampling rate is $L\sigma$, which is the total bandwidth of $x(t)$. We will show that these L sets of uniform samples can be used to reconstruct $x(t)$.

In the ℓth channel, $y_\ell(t)$ contains the samples $x(nT+d_\ell)$; $Y_\ell(\nu)$ consists of shifted versions of $X(\nu)$,

$$Y_\ell(\nu) = \frac{1}{T}\sum_k X(\nu - k\sigma)e^{-jkd_\ell\sigma}.$$

Because the total bandwidth of $x(t)$ is $L\sigma$ and the sampling rate is σ in each channel, serious aliasing occurs in $y_\ell(t)$. To be more quantitative, we first partition the support of $X(\nu)$ into L non overlapping aliasfree(T) sets, $\{S_i\}_{i=0}^{L-1}$. Define $X_i(\nu)$ to be the part of $X(\nu)$ on S_i, i.e., $X_i(\nu) = \begin{cases} X(\nu), & \nu \in S_i \\ 0, & \text{otherwise.} \end{cases}$ On the set S_i, $Y_\ell(\nu)$ contains $X_i(\nu)$ and $L-1$ shifted copies, one from each $X_m(\nu)$, $m \neq i$. Suppose the shifted amounts are respectively $\beta_1(\nu)\sigma$, $\beta_2(\nu)\sigma$, $\ldots$, $\beta_{L-1}(\nu)\sigma$. Because $X_i(\nu)$ are non overlapping, it can be verified that for $\nu \in Supp\{X(\nu)\}$,

$$Y_\ell(\nu) = \frac{1}{T}\left(X(\nu) + \sum_{i=1}^{L-1} X(\nu - \beta_i(\nu)\sigma)e^{-j\beta_i(\nu)d_\ell\sigma}\right).$$

Notice that $\beta_i(\nu)$ thus defined are piecewise constant because $Supp\{X(\nu)\}$ is the union of finitely many intervals.

Lemma 1 [8]. A $U(T,L)$ signal $x(t)$ can be recovered from its PNS(L) samples if and only if the equation below has a solution $\mathbf{f}(\nu)$ for every $\nu \in Supp\{X(\nu)\}$.

$$\mathbf{A}(\nu)\mathbf{f}(\nu) = T\mathbf{e}_0 \quad (1)$$

where $\mathbf{A}(\nu)$ is

$$[\mathbf{A}(\nu)]_{0\ell} = [\mathbf{A}(\nu)]_{\ell 0} = 1, \quad 0 \le \ell \le L-1$$
$$[\mathbf{A}(\nu)]_{i\ell} = e^{-j\beta_i(\nu)d_\ell\sigma}, \quad 1 \le i,\ell \le L-1$$

$\mathbf{f}(\nu) = [F_0(\nu) \quad F_1(\nu) \quad \ldots F_{L-1}(\nu)]^T$ and the vector $\mathbf{e}_0$ is $[1 \quad 0 \quad \ldots \quad 0]^T$. ∎

A nonsingular $\mathbf{A}(\nu)$ will yield unique solutions for the synthesis filters $F_\ell(\nu)$. If we choose $d_\ell = \ell d_1$, $\mathbf{A}(\nu)$ becomes a Vandermonde matrix; the nonsingularity condition becomes much more tractable.

Theorem 1 [8]. Consider a $U(T,L)$ signal $x(t)$. There always exist constant d_ℓ, $0 < \ell < L$ and synthesis filters $F_\ell(\nu)$, $0 \le \ell < L$ such that $x(t) = \sum_{\ell=0}^{L-1}\sum_n x(nT+d_\ell)f_\ell(t-nT)$ (with $d_0 = 0$ in this expression). In particular, the choice

$$d_\ell = \ell d_1, \quad \ell = 1,2,\cdots,L-1,$$

leads to a Vandermonde $\mathbf{A}(\nu)$, which is nonsingular if

$$d_1 \neq \frac{nT}{\beta_i(\nu)} \quad \text{and} \quad d_1 \neq \frac{nT}{(\beta_i(\nu)-\beta_m(\nu))}, i \neq m,$$

for all integer n. The existence of such d_1 is guaranteed. In this case, $\mathbf{f}(\nu) = [F_0(\nu) \ \ldots \ F_{L-1}(\nu)]^T$ is given by

$$\mathbf{f}(\nu) = \begin{cases} T\mathbf{A}(\nu)^{-1}\mathbf{e}_0, & \nu \in Supp\{X(\nu)\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{e}_0 = [1 \quad 0 \quad \ldots \quad 0]^T$. ∎

Remark. The synthesis filters thus obtained are functions of $\beta_i(\nu)$ and can be verified to be piecewise constant [8].

Two-dimensional (2D) case. The aliasfree($\mathbf{T}$) property and aliasfree($\mathbf{T}$) zones can be defined as in 1D case. But now the sampling period $\mathbf{T}$ is a 2×2 nonsingular matrix and the samples are located on the lattice defined by $\mathbf{T}$, i.e., located at $\mathbf{Tn}$ for all integer vectors $\mathbf{n}$. It can be shown that 2D $U(\mathbf{T},L)$ signals can be recovered from PNS(L) samples, $\mathbf{Tn}$, $\mathbf{Tn}+\mathbf{d}_1, \ldots, \mathbf{Tn}+\mathbf{d}_{L-1}$. A result similar to that presented in Theorem 1 can be derived.

3. ONE-DIMENSIONAL DISCRETE-TIME PERIODICALLY NONUNIFORM SAMPLING

In discrete-time PNS(L) sampling (Fig. 4) the total amount of data after decimation is L/M times the original input; the nonuniform sampling scheme makes sense only for $L < M$, which will be assumed throughout this paper. In 1D continuous-time case, we saw that the class $U(T,L)$ allows reconstruction from PNS(L) samples. In this section, a parallel theorem for 1D discrete-time $U(M,L)$ signals will be developed.

The signal $y_\ell(n)$ is the polyphase component $x(nM+d_\ell)$ and $Y_\ell(\omega)$ can be expressed in terms of shifts of $X(\omega)$,

$$Y_\ell(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X(\omega - k\frac{2\pi}{M})e^{-j\frac{2\pi}{M}kd_\ell}.$$

The $U(M,L)$ nature of $X(\omega)$ implies that only L terms in the above summation are nonzero. In particular, on the support of $X(\omega)$, $X(\omega)$ and $L-1$ shifted copies of $X(\omega)$ are nonzero. Let us denote these shifted copies by $X(\omega - \frac{2\pi}{M}\beta_i(\omega))$, $i = 1,2,\cdots,L-1$.

Lemma 2. A 1D discrete-time $U(M,L)$ signal $x(n)$ can be recovered from L of its polyphase components if and only if the equation to follow has a solution for every $\nu \in Supp\{X(\omega)\}$.

$$\mathbf{A}(\omega)\,[F_0(\omega) \quad F_1(\omega) \quad \ldots \quad F_{L-1}(\omega)]^T = T\mathbf{e}_0, \quad (2)$$

where $e_0^T = [1 \;\; 0 \;\; \ldots \;\; 0]$ and the matrix $A(\omega)$ is given by

$$[A(\omega)]_{0\ell} = [A(\omega)]_{\ell 0} = 1, \quad 0 \le \ell \le L-1$$

$$[A(\omega)]_{i\ell} = e^{-j\frac{2\pi}{M}\beta_i(\omega)d_\ell}, \quad 1 \le i, \ell \le L-1 \quad ■$$

Observe that the matrix $A(\omega)$ is a $L \times L$ sub-matrix of the $M \times M$ DFT matrix, W_M given by $[W_M]_{mn} = e^{-j\frac{2\pi}{M}mn}$, $0 \le m,n < M$. Notice that any $L \times L$ sub-matrix of W_M obtained by retaining the first L columns of W_M and some L rows of W_M is a nonsingular Vandermonde matrix. So the choice $d_\ell = \ell$, $\ell = 1, 2, \cdots, L-1$, leads to a nonsingular Vandermonde $A(\omega)$. Unique solutions of $\{F_\ell(\omega)\}$ can be obtained from (2). The theorem below follows.

Theorem 2 [8]. A 1D discrete-time $U(M, L)$ signal $x(n)$ can be recovered from its first L polyphase components, $x(Mn)$, $x(Mn+1)$, ..., $x(Mn+L-1)$.■

4. TWO-DIMENSIONAL SAMPLING AND RECONSTRUCTION

For 2D discrete-time signals, aliasfree(M) property, aliasfree(M) zone and $U(\mathbf{M}, L)$ can be defined in the same manner, where M is a 2×2 nonsingular integer matrix. Similar to 1D case, a necessary and sufficient condition for reconstructing 2D $U(\mathbf{M}, L)$ signals can be derived.

Lemma 3. A 2D discrete-time $U(\mathbf{M}, L)$ signal $x(\mathbf{n})$ can be recovered from L of its polyphase components if and only if the equation to follow has a solution for every $\boldsymbol{\omega} \in Supp\{X(\boldsymbol{\omega})\}$.

$$\mathbf{A}(\boldsymbol{\omega})\,[F_0(\boldsymbol{\omega}) \quad F_1(\boldsymbol{\omega}) \quad \ldots \quad F_{L-1}(\boldsymbol{\omega})]^T = T\mathbf{e}_0, \quad (3)$$

where the matrix $\mathbf{A}(\boldsymbol{\omega})$ is given by

$$[\mathbf{A}(\boldsymbol{\omega})]_{0\ell} = [\mathbf{A}(\boldsymbol{\omega})]_{\ell 0} = 1, \quad 0 \le \ell \le L-1$$

$$[\mathbf{A}(\boldsymbol{\omega})]_{i\ell} = e^{-j2\pi\boldsymbol{\beta}_i^T(\boldsymbol{\omega})\mathbf{M}^{-1}\mathbf{d}_\ell}, \quad 1 \le i, \ell \le L-1 \quad ■$$

In 1D case, we can always choose d_ℓ such that $A(\omega)$ is a nonsingular Vandermonde matrix for every $\omega \in Supp\{X(\omega)\}$. However, it is not always possible to do so in 2D case. In fact, the above equation may not have a solution for some $\boldsymbol{\omega} \in Supp\{X(\boldsymbol{\omega})\}$ and hence $x(\mathbf{n})$ can not be reconstructed from L of its polyphase components. To explain this, we take a closer look at $A(\omega)$.

The matrices $\mathbf{A}(\boldsymbol{\omega})$ and $\mathbf{W}^{(g)}$. It can be verified that $\mathbf{A}(\boldsymbol{\omega})$ is a $L \times L$ sub-matrix of a $J_\mathbf{M} \times J_\mathbf{M}$ matrix $\mathbf{W}^{(g)}$, called the generalized DFT matrix (possibly with some row and column exchanges), where $J_\mathbf{M} = |\det \mathbf{M}|$. The elements of $\mathbf{W}^{(g)}$ are given by

$$[\mathbf{W}^{(g)}]_{in} = e^{-j2\pi\mathbf{k}_i^T\mathbf{M}^{-1}\mathbf{m}_n}, \mathbf{m}_n \in \mathcal{N}(\mathbf{M}), \mathbf{k}_i \in \mathcal{N}(\mathbf{M}^T),$$

where $\mathcal{N}(\mathbf{M})$ denotes the set of integer vectors of the form $\mathbf{Mx}$, $\mathbf{x} \in [0 \;\; 1)^2$. Let $\boldsymbol{\Lambda}$ be the Smith form of $\mathbf{M}$ [6], $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix}$. It can be shown that when $\mathbf{m}_n$ and $\mathbf{k}_i$ are properly ordered, $\mathbf{W}^{(g)} = \mathbf{W}_{\lambda_0} \otimes \mathbf{W}_{\lambda_1}$, where $\mathbf{W}_\lambda$ denotes a $\lambda \times \lambda$ DFT matrix and $\otimes$ denotes the Kronecker product. The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\underbrace{\mathbf{A}}_{I\times K} \otimes \underbrace{\mathbf{B}}_{J\times L} = \underbrace{\begin{bmatrix} a_{0,0}\mathbf{B} & \ldots & a_{0,K-1}\mathbf{B} \\ \vdots & \ldots & \vdots \\ a_{I-1,0}\mathbf{B} & \ldots & a_{I-1,K-1}\mathbf{B} \end{bmatrix}}_{IJ\times KL}.$$

Unlike 1D DFT matrices, $\mathbf{W}^{(g)}$ is not Vandermonde in general; nor are its $L \times L$ sub-matrices obtained by retaining the first L columns and some L rows. The natural question to ask next is whether a particular set of $\{\mathbf{d}_\ell\}$ will make $\mathbf{A}(\boldsymbol{\omega})$ nonsingular for all $\boldsymbol{\omega} \in Supp\{X(\boldsymbol{\omega})\}$. In terms of the generalized DFT matrix $\mathbf{W}^{(g)}$, the question can be recast as follows: can we find L columns of $\mathbf{W}^{(g)}$ such that for arbitrarily chosen L rows of $\mathbf{W}^{(g)}$, the resulting sub-matrix is always nonsingular? The answer is unfortunately, no. Although for every $\boldsymbol{\omega}_0 \in Supp\{X(\boldsymbol{\omega})\}$, there always exist $\{\mathbf{d}_\ell\}$ such that $\mathbf{A}(\boldsymbol{\omega}_0)$ is nonsingular, the same $\mathbf{d}_\ell$ may yield a singular $\mathbf{A}(\boldsymbol{\omega}_1)$ for a different frequency vector $\boldsymbol{\omega}_1$. To follow is an example which demonstrates that there are cases when (3) is not solvable with frequency independent $\{\mathbf{d}_\ell\}$.

Example 1. Consider a discrete-time 2D $U(\mathbf{M}, 2)$ signal $x(\mathbf{n})$, where $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $|\det \mathbf{M}| = 4$. The four vectors in $\mathcal{N}(\mathbf{M})$ are

$$\mathbf{n}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{n}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

Order $\mathbf{k}_i \in \mathcal{N}(\mathbf{M}^T)$ by letting $\mathbf{k}_i = \mathbf{n}_i$, then $\mathbf{W}^{(g)}$ is

$$\mathbf{W}^{(g)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad (4)$$

The support of $X(\boldsymbol{\omega})$, as shown in Fig. 5, consists of two aliasfree(M) zones, S_0 and S_1 with S_1 being the union of three regions R_0, R_1, and R_2. Because $L = 2$, we only have one beta function, $\boldsymbol{\beta}(\boldsymbol{\omega})$. Observe that

$$\boldsymbol{\beta}(\boldsymbol{\omega}) = \begin{cases} \mathbf{k}_1, & \boldsymbol{\omega} \in R_0 \\ \mathbf{k}_2, & \boldsymbol{\omega} \in R_1 \\ \mathbf{k}_3, & \boldsymbol{\omega} \in R_2. \end{cases}$$

So for $\omega \in R_0$, $\mathbf{A}(\omega)$ is a sub-matrix of $\mathbf{W}^{(g)}$ obtained by first keeping the 0th and 1st rows of $\mathbf{W}^{(g)}$ and two columns $\mathbf{W}^{(g)}$. That is, $\mathbf{A}(\omega)$ is a 2×2 sub-matrix of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ obtained by keeping two columns. As to which two columns depends on the choice of $\mathbf{d}_1$. For $\omega \in R_0$, (3) has a solution only if $\mathbf{d}_1$ is $\mathbf{n}_1$ or $\mathbf{n}_3$. If (3) has a solution in each R_i, then

$$\mathbf{d}_1 = \begin{cases} \mathbf{n}_1 \text{ or } \mathbf{n}_3, & \omega \in R_0 \\ \mathbf{n}_2 \text{ or } \mathbf{n}_3, & \omega \in R_1 \\ \mathbf{n}_1 \text{ or } \mathbf{n}_2, & \omega \in R_2. \end{cases}$$

There is no common solution of $\mathbf{d}_1$ for the three regions; (3) does not have a solution for all ω in the support of $X(\omega)$. Therefore $x(\mathbf{n})$ cannot be reconstructed from two of its polyphase components.

A subclass of $U(\mathbf{M}, L)$. Although it is not always possible to reconstruct a $U(\mathbf{M}, L)$ signal from L of its polyphase components, it is always possible to do so when the Smith form $\mathbf{\Lambda}$ of $\mathbf{M} = \mathbf{U\Lambda V}$ is

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & J_{\mathbf{M}} \end{bmatrix}.$$

In this case the generalized DFT matrix $\mathbf{W}^{(g)}$ is the $J_{\mathbf{M}} \times J_{\mathbf{M}}$ DFT matrix $\mathbf{W}_{J_{\mathbf{M}}}$. Similar to the reconstruction of 1D $U(M, L)$ signals, choose $\mathbf{d}_\ell = \ell\mathbf{U}[0 \quad 1]^T$. Then the matrix $\mathbf{A}(\omega)$ will be nonsingular for all $\omega \in Supp\{X(\omega)\}$ and by (3) we can invert $\mathbf{A}(\omega)$ to obtain the synthesis filters.

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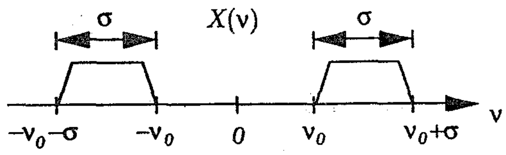

Fig. 1 Bandlimited signal with two bands and total bandwidth $\Theta = 2\sigma$.

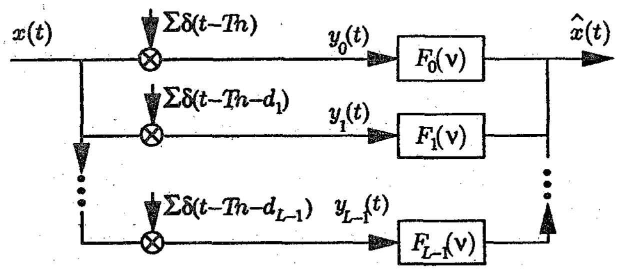

Fig. 2 Illustration of Lth order periodically nonuniform sampling.

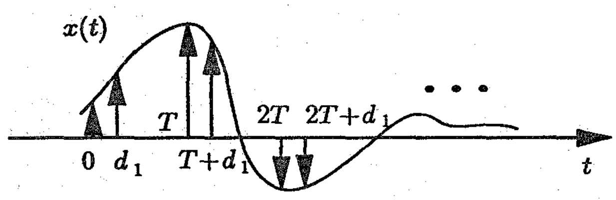

Fig. 3 Reconstruction by using second order periodically nonuniform sampling.

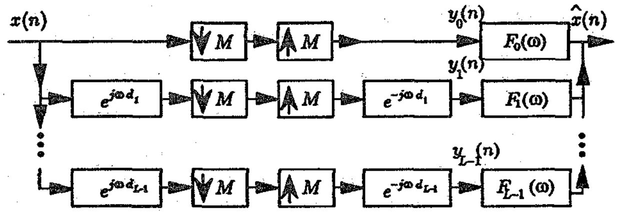

Fig. 4 Periodically nonuniform sampling and reconstruction in discrete-time case.

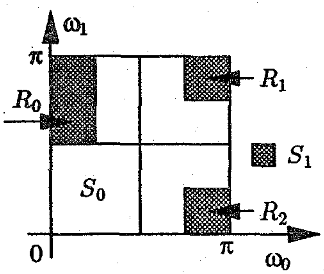

Fig. 5 —Example 1. A $U(\mathbf{M}, 2)$ signal that cannot be reconstructed from two of its polyphase components.