WINDOWED MULTICARRIER SYSTEMS WITH MINIMUM SPECTRAL LEAKAGE

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ABSTRACT

The transmitter output of OFDM or DMT systems with a rectangular window is known to have large spectral sidelobes. Windowing and pulse shaping have been proposed to reduce spectral leakage in the literature. If no extra cyclic prefix is available, windowing at the transmitter requires additional post-processing equalization at the receiver. In this paper we will design optimal windows with minimum spectral leakage. Moreover, we will show that the post processing will affect SNR at the receiver and the resulting SNR can be given in a closed form. Furthermore, we will demonstrate we can have a good trade-off between SNR and spectral leakage.

1. INTRODUCTION

The DFT based multicarrier system has found applications in a wide range of transmission systems, e.g., DMT for ADSL, VDSL, and OFDM for wireless local area network, digital audio broadcasting [1][2]. In the conventional DFT based multicarrier system the pulse shaping filter is a rectangular window. As the rectangular window has large spectral sidelobes, there is a large spectral leakage. This could pose a problem in some applications, where the PSD of the transmit signal is required to have a large enough roll-off in certain frequency bands. For example in some wired transmission application, the PSD of the downstream transmit signal needs to fall below a threshold in the frequency bands of upstream transmission to avoid interference and the PSD should also be attenuated in amateur radio bands to allow egress emission control [2].

Many methods have been proposed to reduce sidelobes by windowing, filtering or using different pulse shaping filters. A number of non-rectangular continuous-time pulse shapes have been proposed to improve the spectral roll-off, e.g., [3]-[5]. Usually continuous-time pulse shapes are designed based on analog implementation of OFDM transmitters and these pulses usually do not admit a digital implementation [6]. Discrete-time windows that can be easily incorporated in digital implementation have been considered in [8, 7] for AWGN channels. In [8], spectrally efficient OFDM systems are designed for AWGN channels using offset QAM. There will be ISI at the output if the channel is not AWGN, but more general ISI channels. In [7], overlapping windows of duration longer than one OFDM symbol is proposed to reduce spectral sidelobes. In this case significant ISI is generated even if the channel is AWGN and post-processing equalizer is needed to remove ISI. If extra guard time is available, post processing can be avoided at the cost of a reduced transmission rate [9]. When there is no extra cyclic prefix, the use of windowing at the transmitter requires post processing at the receiver. More recently, transmitting windows with the cyclic-prefixed property have been proposed in [11]. Windows that are the inverse of a raised cosine function are optimized to minimize spectral leakage and hence minimize egress emission. The corresponding zero-forcing receiver also requires post-processing equalization.

In this paper we will derive the explicit dependency of the post processing on the channel when windowing is applied at the transmitter. We will show that the zero-forcing post processing at the receiver is channel independent if and only if the window itself has the cyclic-prefixed property. We will design optimal window subject to cyclic-prefixed condition to minimize spectral leakage. We will also show that the added post processing will affect SNR at the receiver. A closed form expression of the resulting SNR will be given. Furthermore, we will demonstrate that we can trade SNR for reduced spectral leakage.

Notations. The notation $\mathbf{A}^\dagger$ denotes transpose-conjugate of $\mathbf{A}$. The notation $\mathrm{diag}\begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{M-1}\end{pmatrix}$ denotes an $M \times M$ diagonal matrix with the k-th diagonal element equal to λ_k.

2. WINDOWED MULTICARRIER SYSTEM

The block diagram of the conventional DFT based transceiver with a rectangular window is as shown in Fig. 1. The input modulation symbols s_k are passed through an M-point IDFT, followed by the parallel to serial (P/S) operation and

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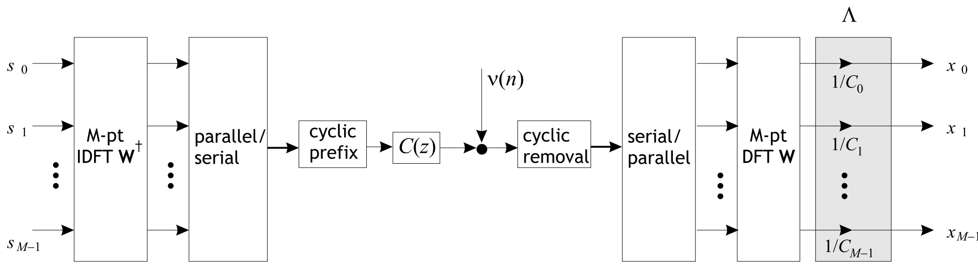

Figure 1: The DFT based transceiver over a channel $C(z)$ with additive noise $\nu(n)$.

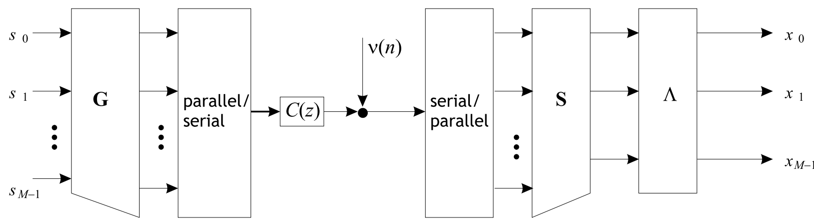

Figure 2: The block based representation of the system in Fig. 2 with transmitting matrix $\mathbf{G}$ and receiving matrix $\mathbf{S}$.

the insertion of cyclic prefix. The length of the cyclic prefix L is chosen to be equal to or larger than the order of the channel $C(z)$. At the receiver, the cyclic prefix is discarded and the samples are again blocked into M by 1 vectors and passed through an $M \times M$ DFT matrix $\mathbf{W}$. The scalar multipliers $1/C_k$ are also called frequency domain equalizers, where $C_0, C_1, \cdots, C_{M-1}$ are the M-point DFT of the channel impulse response c_n. The prefix is discarded at the receiver to remove inter-block ISI. The transceiver is ISI free and the receiver is zero forcing.

The system in Fig. 1 can be redrawn as in Fig. 2. The matrices $\mathbf{G}$ and $\mathbf{S}$ shown in Fig. 2 are of dimensions $N \times M$ and $M \times N$, where $N = M + L$, given respectively by

$$\mathbf{G} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_L \\ \multicolumn{2}{c}{\mathbf{I}_M} \end{pmatrix} \mathbf{W}^\dagger, \quad \text{and} \quad \mathbf{S} = \mathbf{W}\begin{pmatrix} \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{1}$$

The matrix $\mathbf{\Lambda}$ indicated in Fig. 2 is diagonal, given by

$$\mathbf{\Lambda} = \operatorname{diag}\begin{pmatrix} 1/C_0 & 1/C_1 & \cdots & 1/C_{M-1} \end{pmatrix}.$$

We can obtain a windowed transmitter by applying a window to each output block as shown in Fig. 3. The length of the window is the same as the block length N. The window has coefficients $d_0, d_1, \cdots, d_{N-1}$. The conventional OFDM system in Fig. 2 can be viewed as having a rectangular window with length N. Due to the non-rectangular window at the transmitter, the receiver needs an additional post processing matrix $\mathbf{P}$ to cancel inter-subchannel ISI. As there is no constraint on the matrix $\mathbf{P}$, there is no loss of generality in considering the receiver of the form shown in Fig. 3. The transmitting matrix can be written as $\mathbf{DG}$, where $\mathbf{D}$ is the diagonal matrix

$$\mathbf{D} = \operatorname{diag}\begin{pmatrix} d_0 & d_1 & \cdots & d_{N-1} \end{pmatrix}. \tag{2}$$

We partition $\mathbf{D}$ as

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_2 \end{pmatrix},$$

where $\mathbf{D}_0$ and $\mathbf{D}_2$ are of dimensions $L \times L$, and $\mathbf{D}_1$ is of dimensions $(M-L) \times (M-L)$. The condition on $\mathbf{P}$ so that the overall system is ISI free is given below (see [10] for a proof).

Lemma 1 *Consider the system with cyclic prefix in Fig. 3. The receiver is zero forcing if and only if the post processing matrix* $\mathbf{P}$ *is given by*

$$\mathbf{P} = \mathbf{W}\left[\mathbf{W}\begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix} + \mathbf{\Lambda W}\begin{pmatrix} \mathbf{0} & \mathbf{C}_2(\mathbf{D}_0 - \mathbf{D}_2) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right]^{-1},$$

where $\mathbf{C}_2$ *is an* L *by* L *lower triangle Toeplitz matrix with the first column given by* $\begin{pmatrix} c_0 & c_1 & \cdots & c_{L-1} \end{pmatrix}^T$.

From the above lemma, we see that the solution of the post processing matrix depends on the window $\mathbf{D}$ as well as the channel. We observe that $\mathbf{P}$ is channel independent if $\mathbf{D}_0 = \mathbf{D}_2$. That is, the window itself has the cyclic-prefixed property. In this case the solution of $\mathbf{P}$ becomes the same

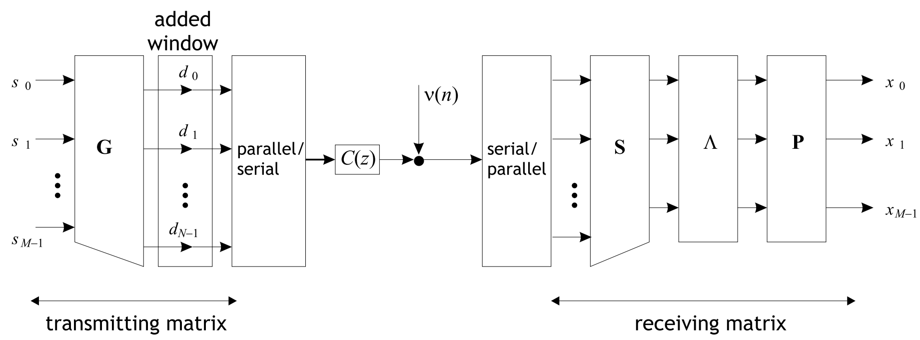

Figure 3: A windowed DFT based transceiver.

as that given in [11]. The resulting post processing matrix is given by

$$\mathbf{P} = \mathbf{W}\text{diag}\begin{pmatrix}1/d_L & 1/d_{L+1} & \cdots & 1/d_{N-1}\end{pmatrix}\mathbf{W}^\dagger. \quad (3)$$

Notice that to have a channel independent $\mathbf{P}$ for any channel, the condition $\mathbf{D}_0 = \mathbf{D}_2$ is not only sufficient but also necessary.

3. SPECTRAL LEAKAGE AND OUTPUT SNR

We showed in Section 2 that a cyclic-prefixed window yields channel independent post processing. We will design windows subject to this constraint. Let $\mathbf{d}$ be the N by 1 window vector and $\widehat{\mathbf{d}} = \begin{pmatrix}d_L & d_{L+1} & \cdots & d_{N-1}\end{pmatrix}$, a vector containing only the last M coefficients of the window. The cyclic-prefixed property means that $\mathbf{d}$ can be written as

$$\mathbf{d} = \mathbf{F}\widehat{\mathbf{d}}, \quad \text{where} \quad \mathbf{F} = \begin{pmatrix}\mathbf{0} & \mathbf{I}_L \\ & \mathbf{I}_M\end{pmatrix}.$$

Let $D(e^{j\omega})$ denote the Fourier transform of the window function, then $D(e^{j\omega}) = \mathbf{e}(e^{j\omega})\mathbf{F}\widehat{\mathbf{d}}$, where $\mathbf{e}(e^{j\omega})$ is the $1\times N$ vector $(1\ e^{-j\omega}\ e^{-j2\omega} \cdots\ e^{-j(N-1)\omega})$. It follows that the squared magnitude response of the window is $|D(e^{j\omega})|^2 = \widehat{\mathbf{d}}^\dagger\mathbf{F}^\dagger\mathbf{e}^\dagger(e^{j\omega})\mathbf{e}(e^{j\omega})\mathbf{F}\widehat{\mathbf{d}}$. Let $\mathbf{E}(e^{j\omega}) = \mathbf{e}^\dagger(e^{j\omega})\mathbf{e}(e^{j\omega})$, then $[\mathbf{E}(e^{j\omega})]_{mn} = e^{j\omega(m-n)}$ and $|D(e^{j\omega})|^2$ can be expressed as

$$|D(e^{j\omega})|^2 = \widehat{\mathbf{d}}^\dagger\mathbf{F}^\dagger\mathbf{E}(e^{j\omega})\mathbf{F}\widehat{\mathbf{d}}.$$

Spectral Leakage. The stopband energy of the window is

$$\mathcal{S} = \int_{\omega_s}^{2\pi-\omega_s} |D(e^{j\omega})|^2 \frac{d\omega}{2\pi}. \quad (4)$$

It can be expressed as

$$\mathcal{S} = \frac{E_s}{N}\widehat{\mathbf{d}}^\dagger\mathbf{F}^\dagger \underbrace{\int_{\omega_s}^{2\pi-\omega_s} \mathbf{E}(e^{j\omega})\frac{d\omega}{2\pi}}_{\mathbf{Q}} \mathbf{F}\widehat{\mathbf{d}} = \widehat{\mathbf{d}}^\dagger\mathbf{F}^\dagger\mathbf{Q}\mathbf{F}\widehat{\mathbf{d}}. \quad (5)$$

The $N \times N$ matrix $\mathbf{Q}$ given in the above equation has the following closed form expression

$$[\mathbf{Q}]_{mn} = \begin{cases} 1 - \frac{\omega_s}{\pi}, & m = n, \\ -\frac{\sin((m-n)\omega_s)}{\pi(m-n)}, & \text{otherwise.} \end{cases}$$

It is real, symmetric and positive semi definite. We define the spectral leakage β as

$$\beta \triangleq \mathcal{S}_d/\mathcal{S}_{rec}, \quad (6)$$

where $\mathcal{S}_{rec}$ is the stopband energy of the rectangular window.

Using (5), we can see that the minimization of spectral leakage becomes the minimization of $\widehat{\mathbf{d}}^\dagger\mathbf{F}^\dagger\mathbf{Q}\mathbf{F}\widehat{\mathbf{d}}$. As the product matrix $\mathbf{F}^\dagger\mathbf{Q}\mathbf{F}$ is positive semi definite, the objective function $\widehat{\mathbf{d}}^\dagger\mathbf{F}^\dagger\mathbf{Q}\mathbf{F}\widehat{\mathbf{d}}$ can be minimized by choosing $\widehat{\mathbf{d}}$ to be the eigen vector corresponding to the smallest eigen value of $\mathbf{F}^\dagger\mathbf{Q}\mathbf{F}$. As the matrix $\mathbf{F}^\dagger\mathbf{Q}\mathbf{F}$ is real, the optimal window also has real coefficients. Notice that the resulting window is different from the solution obtained in [11]. In [11], the window is derived subject to the constraint that the window is the inverse of a raised cosine function.

Output SNR. We assume that the window has the cyclic-prefixed property and the post processing matrix is channel independent as given in (3). Suppose the channel noise $\nu(n)$ is AWGN with spectral density $\mathcal{N}_0$. We constrain the transmission power to be the same as that with a rectangular window. That is, the window satisfies the condition

$$\frac{1}{N}\sum_{k=0}^{N-1} |d_k|^2 = 1. \quad (7)$$

It can be shown that the total output noise power $\mathcal{E}_d = \sum_{k=0}^{M-1} E[|x_k - s_k|^2]$ is given by [10]

$$\mathcal{E}_d = \frac{\mathcal{N}_0}{M}\left[\sum_{i=0}^{M-1}\frac{1}{|C_i|^2}\right]\left[\sum_{k=0}^{M-1}\frac{1}{|d_k|^2}\right]. \quad (8)$$

When the window is rectangular with $d_k = 1$, for all k. The total output noise power in this case is simply $\mathcal{E}_{rec} = \mathcal{N}_0 \sum_{i=0}^{M-1} 1/|C_i|^2$. We define the quantity SNR loss as $\alpha = \mathcal{E}_d/\mathcal{E}_{rec}$. Using (8), we have

$$\alpha = \frac{1}{M} \sum_{k=0}^{M-1} 1/|d_k|^2. \tag{9}$$

We will see later in our experiments that SNR loss α is usually larger than one; windows with better roll-off usually come at the price of a larger SNR loss. Notice that the constraint in (7) is on the average energy of N coefficients ($1/N \sum_{k=0}^{N-1} |d_k|^2 = 1$). If the constraint is on M coefficients, i.e., $1/M \sum_{k=0}^{M-1} |d_k|^2 = 1$, the SNR loss is always greater than one. This is because the function $1/x$ is convex, which implies

$$\frac{1}{M} \sum_{k=0}^{M-1} \frac{1}{|d_k|^2} \geq \frac{1}{\frac{1}{M}\sum_{k=0}^{M-1} |d_k|^2} = 1. \tag{10}$$

Example. The block size $M = 512$ and prefix length $L = 32$. Fig. 4 shows the spectral leakage β and SNR loss α as a function of stopband edge of the window ω_s. As ω_s increases, we can see that the spectral leakage decreases while SNR loss increases. Therefore we can trade SNR for reduced spectral leakage. In particular, when $\omega_s = 1.8\pi/M$, we have $\beta = 0.24$ with $\alpha = 1.58$. Fig. 5 shows the spectrum of the transmitter output when the window is designed using $\omega_s = 1.8\pi/M$. The subcarriers used are 38 to 99 and 111 to 255 as in [11]. We see that the spectrum of the windowed output has a much smaller spectral leakage in unused bands.

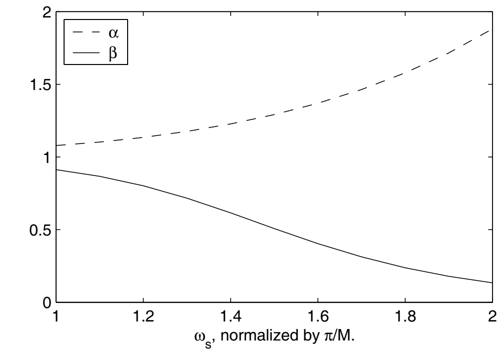

Figure 4: Plot of spectral leakage β and SNR loss α as a function of stopband edge ω_s.

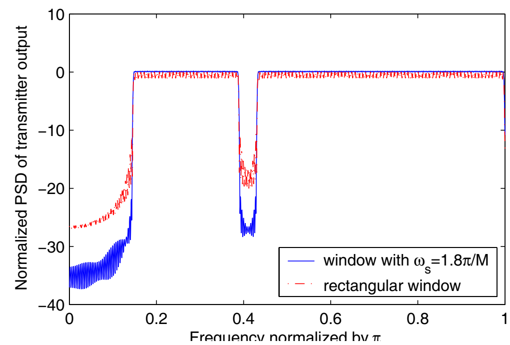

Figure 5: Comparison of the transmitter output power spectrum.

4. CONCLUSIONS

In this paper, we considered window design for multicarrier transmission. The spectral leakage of the transmitter output can be reduced significantly by using windows. The use of windows at the transmitter side requires post processing at the receiver side. Although post processing can lead to an SNR loss at the receiver as we have shown, there is a good trade-off between spectral leakage and output SNR.

5. REFERENCES

[1] ISO/IEC, IEEE Std. 802.11a, 1999.

[2] "Very-high-data-rate digital subscriber line (VDSL) metallic interface," T1.424, 2002.

[3] A. Vahlin and N. Holte, "Optimal Finite Duration Pulses for OFDM," *IEEE Trans. Communications*, vol. 44, no. 1, pp. 10-14, Jan. 1996.

[4] H. Nikookar and R. Prasad, "Optimal waveform design for multicarrier transmission through a multipath channel," Proc. IEEE Vehi. Tech. Conf., May 1997.

[5] S. B. Slimane, "Performance of OFDM systems with time-limited waveforms over multipath radio channels," Global Telecommunications Conference, 1998.

[6] Yuan-Pei Lin and See-May Phoong, "OFDM Transmitters: Analog Representation and DFT Based Implementation," IEEE Trans. Signal Processing, Sep. 2003.

[7] R. W. Lowdermilk, "Design and performance of fading insensitive orthogonal frequency division multiplexing (OFDM) using polyphase filtering techniques, Thirtieth Asilomar Conference on Signals, Systems and Computers, Nov. 1996.

[8] H. Boelcskei, P. Duhamel, and R. Hleiss, "Design of pulse shaping OFDM/OQAM systems for high datarate transmission over wireless channels," Proc. IEEE ICC, 1999.

[9] M. Pauli and P. Kuchenbecker, "On the reduction of the out-of-band radiation of OFDM-signals," Proc. IEEE ICC, 1998.

[10] Yuan-Pei Lin and See-May Phoong, "Window Designs for DFT based Multicarrier Systems," submitted to IEEE Trans. Signal Processing.

[11] G. Cuypers, K. Vanbleu, G. Ysebaert, M. Moonen, "Egress reduction by intra-symbol windowing in dmt-based transmitters," IEEE Proc. Acoustics, Speech, and Signal Processing, 2003.