ANALOG REPRESENTATION AND DIGITAL IMPLEMENTATION OF OFDM SYSTEMS

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ABSTRACT

Many existing results on the analysis of OFDM systems are based on an analog representation. The actual implementation of OFDM transmitters typically consists of a discrete DFT matrix and a digital-to-analog (DAC) converter. In this paper, we show that the analog representation admits the implementation of a DFT matrix followed by a DAC converter only in special cases. Necessary and sufficient conditions for such cases will be given. Analyses based on the analog representation, e.g., spectral roll-off of transmitter outputs and carrier frequency offset, may not be appropriate when digital implementation is employed.

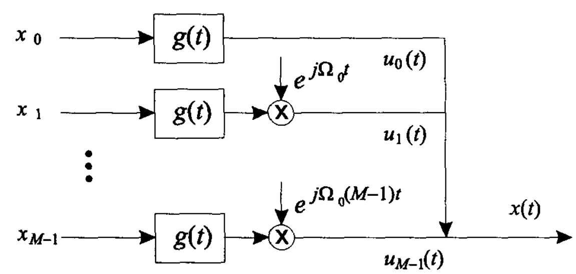

Figure 1: The baseband analog representation of the OFDM transmitter with M subcarriers and pulse shaping filter $g(t)$.

1. INTRODUCTION

The OFDM (orthogonal frequency division multiplexing) systems [2] are well-known for applications in wireless LAN (local area network) and broadcast of digital audio and digital video. Fig. 1 shows the schematic of an analog OFDM transmitter with M subcarriers. Assuming that the subcarrier spacing $= \Omega_0$, the output of the transmitter is given by

$$x(t) = \sum_{k=0}^{M-1} x_k g(t) e^{jk\Omega_0 t}. \quad (1)$$

The pulse shaping filter $g(t)$ is usually the rectangular pulse,

$$g(t) = \begin{cases} 1, & 0 \le t \le T_0, \\ 0, & \text{otherwise}, \end{cases} \quad \text{where } T_0 = 2\pi/\Omega_0 \quad (2)$$

Many studies on OFDM systems are carried out using the expression in (1), e.g., the spectral roll-off of the outputs of OFDM transmitters [3][4], the effect of carrier frequency offset [5], and crest factors of the transmitter outputs [6]. A number of non-rectangular pulse shaping $g(t)$ has been proposed to improve the spectral roll-off of the transmitted signal $x(t)$, e.g., [4][6]. Optimal pulses for minimizing interference is considered in [7] and in [8] parameters of Gaussian pulses are optimized to minimize bit error rates. Although the representation in Fig. 1 is convenient for analysis, the modulation of subcarriers is typically done in the discrete time. Such a transmitter (Fig. 2) consists of two parts [3]: a DAC (digital to analog converter) and the part performing digital modulation of subcarriers, which can be efficiently implemented using an M by M IDFT matrix. The sampling period is $T_s = T_0/M$ and the discrete sequence $w[n]$ shown in Fig. 2 is typically the rectangular window,

$$w[n] = \begin{cases} 1, & 0 \le n \le M-1, \\ 0, & \text{otherwise}, \end{cases} \quad (3)$$

In Chapter 5 of [3], it is mentioned that when we use the digital implementation with an ideal lowpass reconstruction filter $h(t)$ in the DAC converter, the shaping filter $g(t)$ is no longer the rectangular pulse. A precise connection between the analog representation and the digital implementation has not been stated earlier in the literature. In this paper we consider the equivalence of the analog representation and the digital implementation of OFDM transmitters in Fig. 1 and Fig. 2. For the commonly considered case of a rectangular pulse $g(t)$, we show that the two transmitters are not equivalent regardless of the choices of the window $w[n]$ and the reconstruction filter $h(t)$ in the DAC. Also, the digital implementation with a rectangular window $w[n]$ and an ideal lowpass $h(t)$ does not have an equivalent analog representa-

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tion in Fig. 1 for any $g(t)$. Given an arbitrary pulse shaping filter $g(t)$, there does not exist an equivalent digital implementation in general. The analysis of OFDM systems based on the analog representation may not be appropriate if the underlying pulse shaping filter does not allow an equivalent digital implementation. The two systems in Fig. 1 and Fig. 2 can be made equivalent in some special cases. A necessary and sufficient condition for such cases will be derived. An example of a set of $g(t)$, $w[n]$ and $h(t)$ that satisfies the condition will be given.

2. OFDM TRANSMITTERS WITH RECTANGULAR PULSE SHAPING

Suppose the reconstruction filter of the DAC is $h(t)$ as indicated in Fig. 2. The output of the DAC with sampling period T_s is given by

$$y(t) = \sum_{n=0}^{M-1} y[n]h(t - nT_s), \tag{4}$$

where $y[n]$ is the input of the DAC as indicated in Fig. 2.

Theorem 1 *Let the OFDM transmitter in Fig. 1 have a rectangular pulse shaping filter $g(t)$ as given in (2) and the transmitter in Fig. 2 have a discrete rectangular pulse $w[n]$ as given in (3). The outputs of the two systems, respectively $x(t)$ and $y(t)$, are not the same for any choice of reconstruction filter $h(t)$.*

Proof: The signal $y[n]$ is given by

$$y[n] = w[n] \sum_{k=0}^{M-1} x_k e^{j\frac{2\pi}{M}kn}.$$

Substituting the above expression of $y[n]$ into (4), we arrive at

$$y(t) = \sum_{k=0}^{M-1} x_k \sum_{n=0}^{M-1} e^{j\frac{2\pi}{M}kn} h(t - nT_s).$$

Comparing this expression with (1), we conclude that $x(t)$ and $y(t)$ are equal for an arbitrary sequence x_k if and only if

$$\sum_{n=0}^{M-1} e^{j\frac{2\pi}{M}kn} h(t - nT_s) = g(t)e^{jk\Omega_0 t}, \tag{5}$$

for $k = 0, 1, \cdots, M-1$. In particular, the above equation is true for $k = 0$ and $k = 1$. When $k = 0$, we have $\sum_{n=0}^{M-1} h(t - nT_s) = g(t)$. Applying Fourier transform on the both sides of the equation and using $G(j\Omega) = e^{-jT_0\Omega/2}\sin(T_0\Omega/2)/\Omega$, we can verify that $h(t)$ is the rectangular pulse of duration T_s, i.e.,

$$h(t) = \begin{cases} 1, & 0 \le t \le T_s, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

When $k = 1$, we have

$$\sum_{n=0}^{M-1} e^{j2n\pi/M} h(t - nT_s) = g(t)e^{j\Omega_0 t}.$$

Let $f(t) = e^{-j2\pi t/T_0}h(t)$. Then the above equation can be written as $\sum_{n=0}^{M-1} f(t - nT_s) = g(t)$. Similarly, this requires $f(t)$ be the rectangular pulse of duration T_s, which contradicts the solution of $h(t)$ obtained for $k = 0$. Therefore (5) can not be satisfied for any reconstruction filter $h(t)$; hence $x(t)$ and $y(t)$ are not the same. △△△

Example 1. Let us consider the typical choice of $g(t)$, $h(t)$ and $w[n]$ described in [3]. The pulse $g(t)$ is chosen to be the rectangular pulse of duration T_0 given in (2), and $w[n]$ the discrete rectangular window of length M given in (3). The reconstruction filter is an ideal lowpass filter

$$H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi/T_s, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The corresponding impulse response is

$$h(t) = \frac{\sin\frac{\pi t}{T_s}}{\frac{\pi t}{T_s}}. \tag{8}$$

We can easily see that the two systems shown in Fig. 1 and Fig. 2 are not equivalent by observing that $x(t)$ has a finite duration T_0 and the duration of $y(t)$ is not finite. Also $y(t)$ is bandlimited while $x(t)$ is not. Notice that $X(j\Omega)$ has most of its energy in the frequency band $(-\Omega_0/2, (M-0.5)\Omega_0)$ as the mainlobe of $G(j\Omega)$ stretches from $-\Omega_0/2$ to $\Omega_0/2$. On the other hand, the energy of $Y(j\Omega)$ is in the frequency band $(-M\Omega_0/2, M\Omega_0/2)$.

With the above choices of $g(t)$, $h(t)$ and $w[n]$, the two transmitters in Fig. 1 and Fig. 2 are not the same. However, the two transmitters has the following connection. When $x(t)$ and $y(t)$ are sampled with sampling period T_s, the samples are the identical. That is, $x(nT_s) = y(nT_s)$ for $n = 0, 1, \cdots, M-1$. To see that, we can verify $x(nT_s) = \sum_{k=0}^{M-1} x_k e^{jkn2\pi/M}$, for $n = 0, 1, \cdots M-1$, are the M-point DFT of x_k. Therefore

$$x(nT_s) = y[n], n = 0, 1, \cdots, M-1.$$

On the other hand, (4) yields $y(nT_s) = \sum_{m=0}^{M-1} y[m]h((n-m)T_s)$. As $h(t)$ is the ideal lowpass given in (8), it is a Nyquist filter with $h(nT_s) = \delta[n]$, we have

$$y(nT_s) = y[n], n = 0, 1, \cdots, M-1.$$

Such a connection means that if the channel $C(j\Omega)$ is deal ($C(j\Omega) = 1$) and the received signals are sampled with sampling period T_s, the received samples are the same using either one of the two transmitters. The relationship may

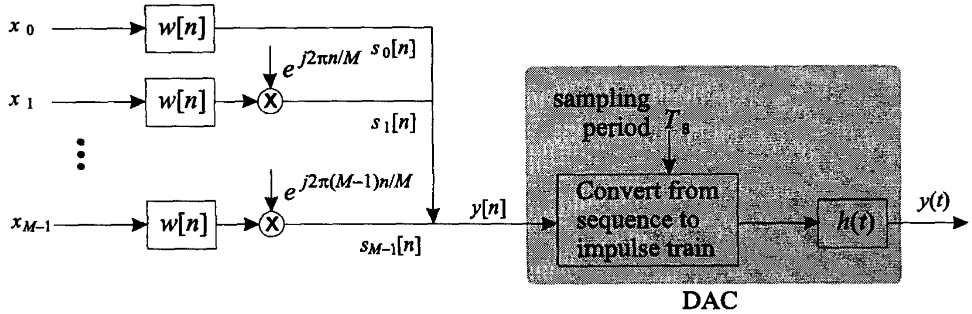

Figure 2: Commonly used digital implementation of the OFDM transmitter.

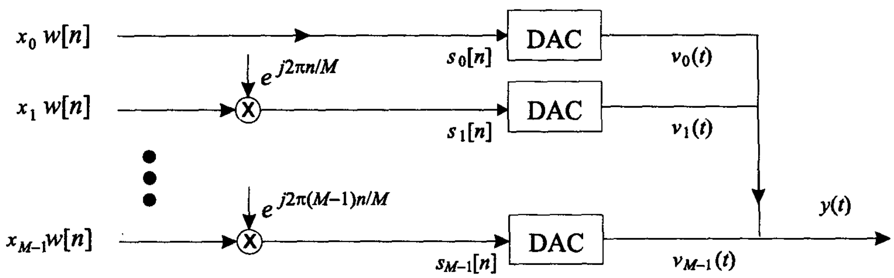

Figure 3: Equivalent block diagram of system in Fig. 2.

not hold for general $C(j\Omega)$. Consider, for example, the case that the channel is a delay, $c(t) = \delta(t - \Delta)$. We can verify that the samples of the received signals for the two transmitters in Fig. 1 and Fig. 2 are different.

3. CONDITIONS FOR EQUIVALENCE OF ANALOG REPRESENTATION AND DIGITAL IMPLEMENTATION

The equivalence of the two systems in Fig. 1 and Fig. 2 can be established in certain cases. For the convenience of derivation, we redraw the system in Fig. 2 as Fig. 3, in which the DAC block is as in Fig. 2. The output due to the k-subcarrier is given by $V_k(j\Omega) = S_k(e^{jT_s\Omega})H(j\Omega)$, [9]. Notice that $S_k(e^{j\omega})$ is a frequency-shifted and scaled version of $W(e^{j\omega})$, i.e.,

$$S_k(e^{j\omega}) = x_k W(e^{j(\omega-2\pi k/M)}), k = 0, 1, \cdots, M-1.$$

Therefore, we have

$$\begin{aligned} V_k(j\Omega) &= x_k W(e^{j(T_s\Omega-2\pi k/M)})H(j\Omega) \\ &= x_k W(e^{jT_s(\Omega-k\Omega_0)})H(j\Omega), \end{aligned} \quad (9)$$

where we have used the facts that $\Omega_0 = 2\pi/T_0$ and $T_0 = MT_s$. On the other hand, the output of the analog representation in Fig. 1 due to the k-th subcarrier is given by

$$U_k(\Omega) = x_k G(j(\Omega - k\Omega_0)).$$

The equivalence of the two systems in Fig. 1 and Fig. 2 means $V_k(\Omega) = U_k(\Omega)$ and therefore

$$W(e^{jT_s(\Omega-k\Omega_0)})H(j\Omega) = G(j(\Omega - k\Omega_0)),$$

for $k = 0, 1, \cdots, M-1$. Summarizing, we have the following theorem.

Theorem 2 *The OFDM transmitter in Fig. 1 can be implemented as in Fig. 2, namely, the two systems are equivalent, if and only if the pulse shaping filter $g(t)$, the digital window $w[n]$ and the reconstruction filter $h(t)$ satisfy:*

$$W(e^{j\Omega T_s})H(j(\Omega + k\Omega_0)) = G(j\Omega), k = 0, 1, \cdots, M-1. \quad (10)$$

In other words, if we are to use a shaping filter $g(t)$ that allows a digital implementation as in Fig. 2, the pulse $g(t)$ should be such that we can find $h(t)$ and $w[n]$ that satisfy (10). Given a discrete window $w[n]$ and a reconstruction filter $h(t)$, there does not exist an equivalent analog representation in general. For example consider the choices of

$h(t)$ and $w[n]$ in Example 1. Using Theorem 2, we can verify that $W(e^{j\Omega T_s})H(j\Omega) \neq W(e^{j\Omega T_s})H(j(\Omega+\Omega_0))$; there does not exist a pulse shaping filter $g(t)$ such that Fig. 1 and Fig. 2 are equivalent. Notice that we did not place any constraint on the duration of $g(t)$, $h(t)$ and $w[n]$ in the derivation; the condition in (10) is valid for non-finite pulses as well.

Corollary 1 *The analog OFDM transmitter with a rectangular pulse $g(t)$ in Fig. 1 does not admit the digital implementation in Fig. 2.*

A proof can be found in [10]. An example of $g(t)$, $w[n]$ and $h(t)$ that meet the requirement in (10) is given in the next example.

Example 2. Consider the case that $g(t)$ is an ideal filter bandlimited to $0 < \Omega < \Omega_0$, as shown in Fig. 4(a),

$$G(j\Omega) = \begin{cases} 1, & 0 < \Omega < \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

We choose $h(t)$ to be an ideal filter of the following frequency characteristics (Fig. 4(b)),

$$H(j\Omega) = \begin{cases} 1, & 0 < \Omega < M\Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

The discrete window $w[n]$ is an ideal filter bandlimited to $0 < \omega < 2\pi/M$ in the period of $0 \leq \omega < 2\pi$. A plot of $W(e^{j\Omega T_s})$ is given in Fig. 4(c). Then $W(e^{j\Omega T_s})$ is periodic with period $2\pi/T_s = M\Omega_0$. In this case we can verify that the condition given in (10) is satisfied and the analog representation and the digital implementation are equivalent.

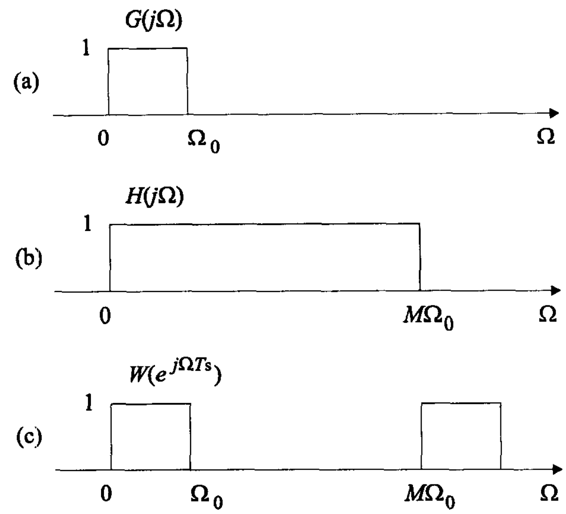

Figure 4: *Example 2.* Illustration of $G(j\Omega)$, $H(j\Omega)$ and $W(e^{j\Omega T_s})$.

4. CONCLUSION

The OFDM transmitter used in practice is based on a DFT matrix followed by a DAC converter. On the other hand the analysis of OFDM systems is usually based on a convenient analog representation with a continuous-time pulse $g(t)$. In this paper, we show that the DFT based OFDM transmitter has an analog representation only in restricted cases. Conversely, given an arbitrary pulse $g(t)$ for the analog transmitter, there is usually no digital DFT based implementation. Therefore designs or analyses of OFDM systems based directly on the digital schematic will be more useful than those based on the analog schematic.

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