LAPPED HADAMARD TRANSFORMS AND FILTER BANKS

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ABSTRACT

In this paper, we generalize the Hadamard transform to the the case of lapped transform. A matrix $\mathbf{A}(z)$ is a lapped Hadamard transform if it satisfies $\mathbf{A}^T(z^{-1})\mathbf{A}(z) = \alpha\mathbf{I}$ for some integer α and all the entries of its coefficient matrices are ± 1. Many methods have been proposed to construct lapped Hadamard matrices. In this paper, we will study these matrices using the theory of paraunitary filter bank. This approach not only greatly simplifies the analysis of lapped Hadamard transform but also gives rise to new construction methods that can generate a much wider class of lapped Hadamard matrices.

1. INTRODUCTION

Hadamard transform has found many applications in various areas of signal processing. An $M \times M$ constant matrix $\mathbf{H}$ is called a Hadamard transform if

$$\mathbf{H}^T\mathbf{H} = M\mathbf{I},$$

and all its entries $h_{ij} \in \{+1, -1\}$. In [1] the author showed that if an $M \times M$ Hadamard matrix exists, then $M = 2$ or M is a multiple of 4. It was widely conjectured that this is also a sufficient condition. In the past, many methods have been introduced for their constructions [2]. This paper studies lapped Hadamard transforms. A causal $M \times M$ polynomial matrix $\mathbf{A}(z) = \sum_{k=0}^{N-1} \mathbf{a}(k)z^{-k}$ is called a lapped Hadamard transform if all the entries of the M by M matrices $\mathbf{a}(k)$ are ± 1 and $\mathbf{A}(z)$ satisfies

$$\mathbf{A}^T(z^{-1})\mathbf{A}(z) = MN\mathbf{I}.$$

When $\mathbf{A}(z)$ is a constant matrix independent of z, the lapped Hadamard matrix reduces to a Hadamard matrix. Matrices satisfying the above expression are also known as paraunitary matrices [3]. In frequency domain, $\mathbf{A}(e^{j\omega})$ is unitary for all frequencies ω. In other words, lapped Hadamard transforms are the special class of paraunitary matrices whose coefficient matrices are antipodal. These matrices are closely related to complementary sequences [4][5]. When a lapped Hadamard matrix $\mathbf{A}(z)$ and its inverse $\mathbf{A}^T(z^{-1})$ are used as the analysis and synthesis polyphase matrices of a filter bank, we have a paraunitary filter bank [3] where all the filter coefficients are ± 1.

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Recently it has been demonstrated in [6] [7] that lapped Hadamard transforms have many potential applications in synchronous spread spectrum communication and CDMA. In the past, many construction methods for lapped Hadamard matrices have been proposed [4][5][6][8][9][10]. In [8] [9], it is shown that we can construct 2×2 lapped Hadamard transforms by cascading 2×2 Hadamard matrices and diagonal matrices with delay elements. Such a method is generalized to the $M \times M$ case in [6] [10]. In [4], the authors show that a 2×2 lapped Hadamard matrix can be constructed from a pair of Golay sequences [11] and vice versa. In [5], the authors show how to construct larger lapped Hadamard matrices from smaller lapped Hadamard matrices. Except [6], all the construction methods are derived using a time-domain approach, which often involves complicated expressions.

In this paper, we apply the theory of paraunitary matrices to the study of lapped Hadamard matrices. All the derivations will be done using a z-domain approach. This approach not only gives a compact description of the previous results, but also enable us to generalize previous methods. Moreover we will introduce some new methods for the construction of lapped Hadamard matrices. The new methods enable us to generate a much wider class of lapped Hadamard matrices.

2. DEFINITIONS AND PRELIMINARIES

In this section, we will describe a number of tools that will be useful for later discussions. Consider the polynomial matrix $\mathbf{A}(z) = \sum_{n=0}^{N-1} \mathbf{a}(n)z^{-n}$ with nonzero $\mathbf{a}(0)$ and $\mathbf{a}(N-1)$. The constant matrices $\mathbf{a}(n)$ are the coefficient matrices. The numbers N and $(N-1)$ are respectively the length and the order of $\mathbf{A}(z)$. All matrices studied in this paper are square matrices. The tilde of $\mathbf{A}(z)$ is defined as

$$\widetilde{\mathbf{A}}(z) = \mathbf{A}^T(z^{-1}).$$

Using the tilde notation, an $M \times M$ matrix $\mathbf{A}(z)$ is **paraunitary (PU)** if

$$\widetilde{\mathbf{A}}(z)\mathbf{A}(z) = c\mathbf{I}, \text{ for some nonzero constant } c. \quad (1)$$

When all the entries of all the nonzero coefficient matrices $\mathbf{a}(n)$ are ± 1, then $\mathbf{A}(z)$ will be called an **antipodal (AP)** matrix. A PU AP matrix will be called a lapped Hadamard matrix.

Let $\mathbf{A}(z)$ and $\mathbf{B}(z)$ be two $M \times M$ AP matrices with lengths N_a and N_b respectively. In general, the AP property is not preserved by additions and multiplications. However it can be verified that the following two matrices are AP:

$$\mathbf{A}(z)\mathbf{B}(z^{N_a}), \ \mathbf{A}(z) + z^{-N_a}\mathbf{B}(z).$$

Moreover if $\mathbf{A}(z)$ and $\mathbf{B}(z)$ have the same length, the matrix $\mathbf{A}(z^2)+z^{-1}\mathbf{B}(z^2)$ will also be AP. These AP-property preserving operations will be useful in understanding of many construction methods described later.

Kronecker product will be useful for the construction of larger lapped Hadamard matrices from smaller lapped Hadamard matrices. Given two square matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ with dimensions M_a and M_b respectively, their Kronecker product $\mathbf{A}(z)\otimes\mathbf{B}(z)$ is defined as (we have dropped the argument z for notational simplicity)

$$\begin{pmatrix} A_{00}\mathbf{B} & A_{01}\mathbf{B} & \cdots & A_{0,M-1}\mathbf{B} \\ A_{10}\mathbf{B} & A_{11}\mathbf{B} & \cdots & A_{1,M-1}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M-1,0}\mathbf{B} & A_{M-1,1}\mathbf{B} & \cdots & A_{M-1,M-1}\mathbf{B} \end{pmatrix},$$

where $A_{ij}(z)$ is the ijth element of $\mathbf{A}(z)$. Note that $\mathbf{A}(z)\otimes\mathbf{B}(z)$ is an $M_aM_b\times M_aM_b$ matrix. Moreover if the lengths of $\mathbf{A}(z)$ and $\mathbf{B}(z)$ are N_a and N_b respectively, then the length of $\mathbf{A}(z)\otimes\mathbf{B}(z)$ will be N_a+N_b-1. One can verify that the tilde of $\mathbf{A}(z)\otimes\mathbf{B}(z)$ is equal to $\widetilde{\mathbf{A}}(z)\otimes\widetilde{\mathbf{B}}(z)$. Let the dimensions of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ be so that all the matrix multiplications in the following expression are defined. Then the product rule states that

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD}).$$

Using the product rule, one can immediately show that given two PU matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ (not necessarily of the same dimensions), their Kronecker product $\mathbf{A}(z)\otimes\mathbf{B}(z)$ is also PU.

3. EXISTING CONSTRUCTION METHODS

In this section, we will review come existing construction methods for lapped Hadamard matrices. Though many of these methods were originally derived using time-domain sequences, we will adopt the z-domain expression as it gives a more compact expression. Moreover, we will use the theory of PU matrices to explain these methods.

It was shown in [4] that 2×2 lapped Hadamard transforms are closely related to complementary sequences, or more commonly known as Golay sequences. A pair of AP sequences $A_i(z) = \sum_{n=0}^{N-1} a_i(n)z^{-n}$ $(i=0,1)$ are complementary if they satisfy [11]

$$A_0(z)\widetilde{A}_0(z) + A_1(z)\widetilde{A}_1(z) = 2N.$$

Using these sequences, we form

$$\mathbf{E}(z) = \begin{pmatrix} A_0(z) & -z^{-N+1}\widetilde{A}_1(z) \\ A_1(z) & z^{-N+1}\widetilde{A}_0(z) \end{pmatrix}.$$

One can verify by direct multiplication that $\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = 2N\mathbf{I}$; the matrix $\mathbf{E}(z)$ is a lapped Hadamard matrix. Though complementary sequences can be generalized to the case of M sequences [5], unfortunately there is no known method to construct $M\times M$ lapped Hadamard matrices from M complementary sequences.

In [8] [9] [6] [10], it was shown that lapped Hadamard transforms can be constructed from Hadamard matrices. Let M be such that Hadamard matrices exist and let $\mathbf{H}_k$ be Hadamard matrices. Let $\mathbf{E}_0(z) = \mathbf{H}_0$. Consider the following $M\times M$ matrices:

$$\mathbf{E}_{k+1}(z) = \mathbf{H}_k\mathbf{\Lambda}(z^{M^k})\mathbf{E}_k(z), \text{ for } k = 0, 1, \ldots, \tag{2}$$

where $\mathbf{\Lambda}(z)$ is the diagonal matrix

$$\mathbf{\Lambda}(z) = diag[1, z^{-1}, \ldots, z^{-M+1}].$$

As both $\mathbf{H}_k$ and $\mathbf{\Lambda}(z)$ are PU (1), so are their products. Hence $\mathbf{E}_k(z)$ are PU for all k. Moreover it is not difficult to verify that all the coefficient matrices in $\mathbf{E}_k(z)$ have ± 1 as their entries. Thus $\mathbf{E}_k(z)$ are lapped Hadamard transforms. Note that the length of the lapped Hadamard matrix $\mathbf{E}_k(z)$ is M^k. For moderate numbers M and k, the length of $\mathbf{E}_k(z)$ becomes very large.

In [5], several algorithms were given for the construction of larger lapped Hadamard matrices from smaller lapped Hadamard matrices. Let $\mathbf{A}(z)$ be an $M\times M$ lapped Hadamard matrix with length N. Then consider the following two $2M\times 2M$ matrices

$$\mathbf{E}_0(z) = \begin{pmatrix} \mathbf{A}(z^2)+z^{-1}\mathbf{A}(z^2) & -\mathbf{A}(z^2)+z^{-1}\mathbf{A}(z^2) \\ -\mathbf{A}(z^2)+z^{-1}\mathbf{A}(z^2) & \mathbf{A}(z^2)+z^{-1}\mathbf{A}(z^2) \end{pmatrix}. \tag{3}$$

$$\mathbf{E}_1(z) = \begin{pmatrix} \mathbf{A}(z)+z^{-N}\mathbf{A}(z) & -\mathbf{A}(z)+z^{-N}\mathbf{A}(z) \\ -\mathbf{A}(z)+z^{-N}\mathbf{A}(z) & \mathbf{A}(z)+z^{-N}\mathbf{A}(z) \end{pmatrix}. \tag{4}$$

Clearly both $\mathbf{E}_i(z)$ are AP matrices with length $2N$. Using the paraunitariness of $\mathbf{A}(z)$, one can verify by direct multiplication that $\mathbf{E}_i(z)$ are also PU. Thus $\mathbf{E}_i(z)$ are lapped Hadamard matrices. The matrices $\mathbf{E}_0(z)$ and $\mathbf{E}_1(z)$ can be viewed as "interlaced" and "cascade" versions of $\mathbf{A}(z)$ respectively. By repeatedly applying the above methods, starting from a 2×2 lapped Hadamard transform, one can generate $2^k\times 2^k$ lapped Hadamard transforms for all integer k.

It was also shown in [5] that we can construct lapped Hadamard matrices by applying the Kronecker product. Let $\mathbf{H}$ be an $M_h\times M_h$ Hadamard matrix and $\mathbf{A}(z)$ be an $M_a\times M_a$ lapped Hadamard matrix with length N_a. Form the following $M_hM_a\times M_hM_a$ matrix

$$\mathbf{E}(z) = \mathbf{H}\otimes\mathbf{A}(z). \tag{5}$$

It is clearly an AP matrix. As the Kronecker product of PU matrices is also PU, $\mathbf{E}(z)$ is a lapped Hadamard matrix of length N_a. Though it was not mentioned in [5], one can verify that $\mathbf{A}(z)\otimes\mathbf{H}$ is also a lapped Hadamard matrix.

4. NEW RESULTS

In the following, we will first generalize the results in (3), (4) and (5). Then two new construction methods will be given.

Let $\mathbf{A}(z)$ and $\mathbf{B}(z)$ be lapped Hadamard matrices with the same dimension M and the same length N_a. Then one can generalize the result in (3) by constructing the matrix

$$\mathbf{E}_0(z) = \begin{pmatrix} \mathbf{A}(z^2)+z^{-1}\mathbf{B}(z^2) & -\mathbf{A}(z^2)+z^{-1}\mathbf{B}(z^2) \\ -\mathbf{A}(z^2)+z^{-1}\mathbf{B}(z^2) & \mathbf{A}(z^2)+z^{-1}\mathbf{B}(z^2) \end{pmatrix}.$$

It is not difficult to verify by direct multiplication that the above $\mathbf{E}_0(z)$ is a lapped Hadamard matrix with length $2N_a$. Let $\mathbf{C}(z)$ be another $M\times M$ lapped Hadamard matrix with length N_c. Then one can verify that the following matrix is a lapped Hadamard transform with length (N_a+N_c).

$$\mathbf{E}_1(z) = \begin{pmatrix} \mathbf{A}(z)+z^{-N_a}\mathbf{C}(z) & -\mathbf{A}(z)+z^{-N_a}\mathbf{C}(z) \\ -\mathbf{A}(z)+z^{-N_a}\mathbf{C}(z) & \mathbf{A}(z)+z^{-N_a}\mathbf{C}(z) \end{pmatrix}.$$

The above result can be viewed as a generalization of (4).

One can also generalize (5) by taking the Kronecker product of two lapped Hadamard matrices. However special care has to be taken so that the AP property of these matrices is not destroyed. Let $\mathbf{B}(z)$ and $\mathbf{C}(z)$ be lapped Hadamard matrices of dimensions M_b and M_c respectively. Let N_b and N_c be their lengths respectively. Then one can show that the resulting matrices of the following two Kronecker products are lapped Hadamard matrices with length $(N_b + N_c - 1)$ and dimension M_bM_c.

$$\mathbf{E}_0(z) = \mathbf{B}(z^{N_c}) \otimes \mathbf{C}(z) \quad (6)$$

$$\mathbf{E}_1(z) = \mathbf{B}(z) \otimes \mathbf{C}(z^{N_b}) \quad (7)$$

The above seemingly simple generalization of the Kronecker product method includes (3), (4) and (5) as special cases. To see this, let

$$\mathbf{B}(z) = \begin{pmatrix} 1+z^{-1} & -1+z^{-1} \\ -1+z^{-1} & 1+z^{-1} \end{pmatrix}, \text{ and } \mathbf{C}(z) = \mathbf{A}(z).$$

Then (6) and (7) reduce respectively to (3) and (4).

Construction Method Using the Agayan-Sarukhanyan Multiplication Theorem: By taking the Kronecker product of two matrices with dimensions M_a and M_b, we will get a matrix of dimension M_aM_b. In [2], it was shown that we can reduce the dimension using the elegant multiplication theorem of Agayan-Sarukhanyan. It was shown that given two Hadamard matrices of dimensions M_a and M_b, one can construct a Hadamard matrix of dimension $M_aM_b/2$. WIt turns out that we can also apply the multiplication theorem of Agayan-Sarukhanyan to the construction of lapped Hadamard matrices. Let $\mathbf{A}(z)$ and $\mathbf{B}(z)$ be lapped Hadamard matrices of dimensions M_a and M_b respectively. Suppose that their lengths are N_a and N_b respectively. Consider the following partitions:

$$\mathbf{A}(z) = \begin{pmatrix} \mathbf{A}_{00}(z) & \mathbf{A}_{01}(z) \\ \mathbf{A}_{10}(z) & \mathbf{A}_{11}(z) \end{pmatrix}, \quad \mathbf{B}(z) = \begin{pmatrix} \mathbf{B}_{00}(z) & \mathbf{B}_{01}(z) \\ \mathbf{B}_{10}(z) & \mathbf{B}_{11}(z) \end{pmatrix},$$

where $\mathbf{A}_{ij}(z)$ and $\mathbf{B}_{ij}(z)$ are $M_a/2 \times M_a/2$ and $M_b/2 \times M_b/2$ matrices respectively. This partition is always possible as M_a and M_b are even (see the remark at the end of this section for a proof). Form the following $\frac{M_aM_b}{2} \times \frac{M_aM_b}{2}$ matrix with length (N_aN_b):

$$\mathbf{C}(z) = \begin{pmatrix} \mathbf{C}_{00}(z) & \mathbf{C}_{01}(z) \\ \mathbf{C}_{10}(z) & \mathbf{C}_{11}(z) \end{pmatrix},$$

where the submatrices are given by

$$\begin{aligned} \mathbf{C}_{00}(z) = {} & \tfrac{1}{2}(\mathbf{A}_{00}(z^{N_b}) + \mathbf{A}_{01}(z^{N_b})) \otimes \mathbf{B}_{00}(z) + \\ & \tfrac{1}{2}(\mathbf{A}_{00}(z^{N_b}) - \mathbf{A}_{01}(z^{N_b})) \otimes \mathbf{B}_{10}(z), \end{aligned}$$

$$\begin{aligned} \mathbf{C}_{01}(z) = {} & \tfrac{1}{2}(\mathbf{A}_{00}(z^{N_b}) + \mathbf{A}_{01})(z^{N_b}) \otimes \mathbf{B}_{01}(z) + \\ & \tfrac{1}{2}(\mathbf{A}_{00}(z^{N_b}) - \mathbf{A}_{01}(z^{N_b})) \otimes \mathbf{B}_{11}(z), \end{aligned}$$

$$\begin{aligned} \mathbf{C}_{10}(z) = {} & \tfrac{1}{2}(\mathbf{A}_{10}(z^{N_b}) + \mathbf{A}_{11}(z^{N_b})) \otimes \mathbf{B}_{00}(z) + \\ & \tfrac{1}{2}(\mathbf{A}_{10}(z^{N_b}) - \mathbf{A}_{11}(z^{N_b})) \otimes \mathbf{B}_{10}(z), \end{aligned}$$

$$\begin{aligned} \mathbf{C}_{11}(z) = {} & \tfrac{1}{2}(\mathbf{A}_{10}(z^{N_b}) + \mathbf{A}_{11}(z^{N_b})) \otimes \mathbf{B}_{01}(z) + \\ & \tfrac{1}{2}(\mathbf{A}_{10}(z^{N_b}) - \mathbf{A}_{11}(z^{N_b})) \otimes \mathbf{B}_{11}(z). \end{aligned}$$

The matrix $\mathbf{C}(z)$, formed in such a manner, is called the Agayan-Sarukhanyan multiplication of $\mathbf{A}(z^{N_b})$ and $\mathbf{B}(z)$, denoted as

$$\mathbf{C}(z) = \mathbf{A}(z^{N_b}) \otimes_{AS} \mathbf{B}(z).$$

One can verify that all $\mathbf{C}_{ij}(z)$ are AP matrices of the same length and hence $\mathbf{C}(z)$ is an AP matrix. Applying the PU properties of $\mathbf{A}(z)$ and $\mathbf{B}(z)$, one can show that $\widetilde{\mathbf{C}}(z)\mathbf{C}(z) = \alpha\mathbf{I}$, where $\alpha = \frac{1}{4}M_a^2M_b^2N_aN_b$. Hence $\mathbf{C}(z)$ is a lapped Hadamard matrix. Clearly, one can verify that $\mathbf{A}(z) \otimes_{AS} \mathbf{B}(z^{N_a})$, where N_a is the length of $\mathbf{A}(z)$, is also a lapped Hadamard transform. Note that when one of the matrices, say $\mathbf{A}(z)$, has dimension $M_a = 2$, then the dimension of $\mathbf{C}(z)$ will be M_b. Using this method, one can generate lapped Hadamard matrix with length 2^k for any integer k. Let M be such that Hadamard matrices $\mathbf{H}$ of dimension M exist. Let

$$\mathbf{E}_1(z) = \begin{pmatrix} 1+z^{-1} & -1+z^{-1} \\ -1+z^{-1} & 1+z^{-1} \end{pmatrix} \otimes_{AS} \mathbf{H}.$$

Clearly $\mathbf{E}_1(z)$ is an $M \times M$ lapped Hadamard matrix of length 2^1. For $k \geq 1$, we carry out the following iterations:

$$\mathbf{E}_{k+1}(z) = \begin{pmatrix} 1+z^{-2^k} & -1+z^{-2^k} \\ -1+z^{-2^k} & 1+z^{-2^k} \end{pmatrix} \otimes_{AS} \mathbf{E}_k(z).$$

Clearly all $\mathbf{E}_k(z)$ are $M \times M$ PU matrices as the Agayan-Sarukhanyan multiplication preserves the PU property. Moreover $\mathbf{E}_k(z)$ are AP matrices with length 2^k. Hence $\mathbf{E}_k(z)$ are $M \times M$ lapped Hadamard matrices with length 2^k. Comparing our results with (2), we see that the matrices constructed using (2) have length equal to M^k whereas our matrices have length 2^k.

Butterfly Structure Method: Let M be a number such that Hadamard matrices exist. From [1], we know that M is either 2 or a multiple of 4. Define the following two $M \times M$ matrices:

$$\begin{aligned} \mathcal{B}_M &= \mathbf{I}_{M/2} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ \boldsymbol{\theta}(z) &= diag[1 \ z^{-1} \ 1 \ z^{-1} \ \cdots \ 1 \ z^{-1}]. \end{aligned}$$

Let $\mathbf{E}_0(z) = \mathbf{H}$, an $M \times M$ Hadamard matrix. For $k \geq 0$, we form

$$\mathbf{E}_{k+1}(z) = \mathcal{B}_M \boldsymbol{\theta}(z^{2^k}) \mathbf{P}_k \mathbf{E}_k(z), \quad (8)$$

where $\mathbf{P}_k$ are $M \times M$ permutation matrices. It is clear that $\mathbf{E}_k(z)$ are PU. Moreover they are also AP matrices due to the insertion of delay elements in $\boldsymbol{\theta}(z)$. The length of $\mathbf{E}_k(z)$ is 2^k. For example, Fig. 1 shows the implementation of $\mathbf{E}_k(z)$ for $k = 2$ and $M = 4$. Note that the butterfly structure has an additional advantage of low complexity. The computational cost for adding one stage is M additions. Note that when M is a power of two, $\mathbf{H}$ can also be realized using $\log_2 M$ stages of the butterflies [12]. To implement a lapped Hadamard matrix of length 2^k, we only need $(k + \log_2 M)M$ additions.

Connection Between the Butterfly Structure Method and (2): When the number of channels M is a power of two, we can show that the butterfly structure method includes (2) as a special case. We demonstrate this for the case $M = 8$. To do this, we need to show that $\mathbf{HA}(z)$ in (2) can be expressed as a product of matrices of the form $\mathcal{B}_8\boldsymbol{\theta}(z)\mathbf{P}$ as in (8). It is well-known [12] that the 8×8 Hadamard matrix can be implemented efficiently the butterflies. Using such an efficient structure, we can implement $\mathbf{HA}(z)$

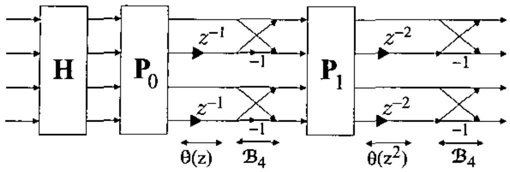

Figure 1: A 4×4 lapped Hadamard matrix $\mathbf{E}_2(z)$ constructed by *the butterfly method.*

as in Fig. 2(a). After moving some delay elements to the right of the butterflies, we can redraw Fig. 2(a) as Fig. 2(b). Note that each stage (indicated by the box) in Fig. 2(a) can be described by $\mathcal{B}_8\boldsymbol{\theta}(z^{2^i})\mathbf{P}_i$ by choosing the permutation matrix $\mathbf{P}_i$ properly.

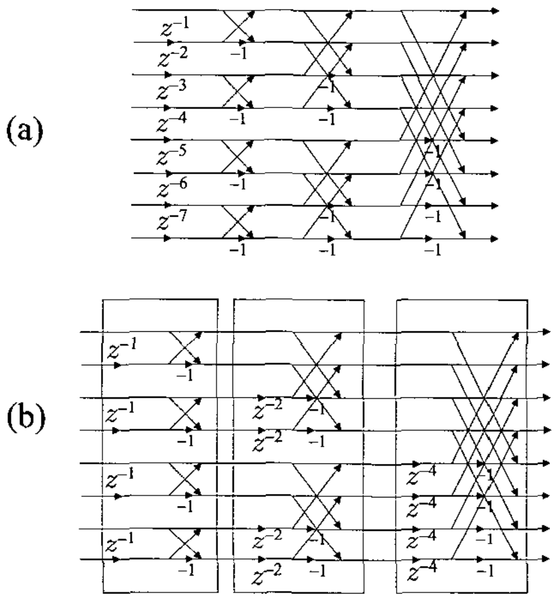

Figure 2: (a) An implementation of $\mathbf{HA}(z)$. (b) An equivalent system.

Remark: It was known [1] that Hadamard matrices $\mathbf{H}$ exist only for dimensions of 2 or a multiple of 4. Whether this is also a necessary condition for the existence of lapped Hadamard matrices is still unknown. But it is easy to see that the dimension of lapped Hadamard matrices has to be even. Too see this, let $\mathbf{A}(z) = \sum_{n=0}^{N-1} \mathbf{a}(n)z^{-n}$ be an $M \times M$ lapped Hadamard transform. The PU property of $\mathbf{A}(z)$ (defined in (1)) implies that

$$\mathbf{a}^T(N-1)\mathbf{a}(0) = \mathbf{0}.$$

As $\mathbf{a}(n)$ are AP matrices, the above equation implies that the dimension M is even.

5. CONCLUSIONS

In this paper, we have studied lapped Hadamard matrices. The theory of PU matrices was applied to analysis and synthesize these matrices. Using such an approach, we can not only prove all previous construction methods in a simple manner but also generalize their results. New methods that can generate a much wider class of lapped Hadamard matrices are also introduced. One can generalize the definition of lapped Hadamard matrix to the complex case. An $M \times M$ matrix $\mathbf{A}(z) = \sum_{n=0}^{N-1} \mathbf{a}(n)z^{-n}$ is a lapped Hadamard matrix if all the entries of the coefficient matrices have unit magnitude and $\mathbf{A}^\dagger(1/z^*)\mathbf{A}(z) = MN\mathbf{I}$, where * and † denote the complex conjugate and transpose conjugate respectively. It can be verified that except the method using Agayan-Sarukhanyan multiplication theorem, all the methods described in this paper can be modified for the construction of complex lapped Hadamard matrices.

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6. REFERENCES

[1] R. E. A. C. Paley, "On orthogonal matrices," *J. Math. Phys.*, vol. 12, pp. 311–320, 1933.

[2] J. Seberry and M. Yamada, "Hadamard Matrices, Sequences and Block Designs," *in Contemporary Design Theory: A Collection of Surveys*, edited by J. H. Dinitz and D. R. Stinson, John Wiley & Sons, Inc., 1992.

[3] P. P. Vaidyanathan, *Multirate systems and filter banks*, Prentice-Hall, 1993.

[4] Y. Taki, H. Miyakawa, M. Hatori and S. Namba, "Even-shift orthogonal sequences," *IEEE Trans. Inform. Theory*, vol. IT-15, pp. 295–300, Mar. 1969.

[5] C.-C. Tseng and C. L. Liu, "Complementary sets of sequences," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 644–652, Sep. 1972.

[6] G. W. Wornell "Emerging applications of mutirate signal processing and wavelets in digital communications," *Proceedings of the IEEE*, vol. 84, pp. 586–1187, Aug. 1996.

[7] H. Chen, J. Yeh, and N. Suehiro, "A multicarrier CDMA architecture based on orthogonal complementary codes for new generations of wideband wireless communications," *IEEE Communications Magazine*, pp. 126–135, Oct. 2001.

[8] H. S. Shapiro, "Extremal problems for polynomials and power series," Master's thesis, MIT, Cambridge, MA, 1951.

[9] W. Rudin, "Some theorems on Fourier coefficients," *Proc. Amer. Math. Soc.*, vol. 10, pp. 855-859, 1959.

[10] N. Suehiro and M. Hatori, "N-shift cross-orthogonal sequences," *IEEE Trans. Information Theory*, vol. 34, no. 1, Jan. 1988.

[11] M. J. E. Golay, "Complementary series," *IRE Trans. Inform. Theory*, vol. IT-7, pp. 82–87, 1961.

[12] N. Ahmed and K. R. Rao, *Orthogonal Transforms for Digital Signal Processing*, Springer-Verlag, 1975.