# ON THE DUALITY OF OPTIMAL DMT SYSTEMS AND BIORTHOGONAL SUBBAND CODERS

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## ABSTRACT

The discrete multitone modulation (DMT) systems have been widely used in various applications. The DMT system can be considered as a dual of a subband coder, obtained by using the synthesis bank as the transmitter and analysis bank as the receiver. In designing optimal subband coders, the objective is to minimize output quantization noise, whereas in the problem of designing optimal DMT system, the objective function to be minimized is the transmitted power. In this paper we will show that the design of optimal DMT systems can be formulated as a hypothetical design problem of optimal subband coders. The solution of optimal DMT system can be obtained using existing design methods for optimal biorthogonal subband coders.

## 1. INTRODUCTION

The discrete multitone modulation (DMT) systems have been shown to be a very useful for transmission over frequency selective channels [1][2][3]. Recently there has been considerable interest in the design of optimal DMT systems [4][5]. Fig. 1 shows an $M$-band DMT system over a frequency selective channel $C(z)$ with additive channel noise $e(n)$. The transmitting and receiving filters are respectively $T_k(z)$ and $R_k(z)$, and the DMT system $\mathcal{D}$ is denoted by $\mathcal{D} = \{T_k(z), R_k(z)\}$. The inputs $x_k(n)$ of the transmitter are modulation symbols, e.g., PAM or QAM symbols. Each symbol of the $k$-th band contains $b_k$ bits. The average bit rate is $b = 1/M\sum_{k=0}^{M-1} b_k$. We say the DMT system is perfect if the outputs $\widehat{x}_k(n) = x_k(n)$, for $k = 0, 1, \cdots, M-1$ in the absence of channel noise $e(n)$. In this case, there is no inter- and intra-band ISI. When there is channel noise, $\widehat{x}_k(n) = x_k(n) + e_k(n)$, where the noise $e_k(n)$ of the $k$-th band comes entirely from the channel noise $e(n)$. For a given average bit rate, the optimal DMT system minimizes the transmitted power $\mathcal{P}$, i.e., the variance of the transmitted signal $y(n)$ as indicated in Fig. 1.

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The $M$-band DMT system can be viewed as a dual of an $M$-band subband coder (Fig. 2) by interchanging the analysis and synthesis bank. The filter bank with analysis filters $H_k(z)$ and synthesis filters $F_k(z)$, denoted by $\mathcal{F} = \{H_k(z), F_k(z)\}$, is said to be biorthogonal or have perfect reconstruction (PR) property if

$$\left(F_k(e^{j\omega})H_m(e^{j\omega})\right)_{\downarrow M} = \delta(k-m),$$

where $\downarrow M$ denotes $M$-fold decimation. When there is quantization noise, the output $\widehat{x}(n) = x(n) + q(n)$, where $q(n)$ comes entirely from the quantization noise $q_k(n)$. A PR filter bank is called orthonormal if $F(e^{j\omega}) = H^*(e^{j\omega})$. For a given class of filter banks, the optimal solution is one that minimizes the output noise variance $\sigma_q^2$.

In the context of optimal subband coder design, great advance has been made recently [6][7][8][9][10]. It has been shown that, for the class of orthonormal filter banks, the Principle Component Filter Bank (PCFB) minimizes the output noise variance $\sigma_q^2$. For the design of biorthogonal filter banks, the structure of cascading orthonormal (ParaUnitary) filter banks with pre- and post-filters (PPU structure) is proposed in [11] to minimize the output noise. Recently, Moulin et. al. show that [10] there is no loss of generality in assuming the PPU structure in the design of optimal biorthogonal filter banks. More recently, it is shown that PCFB is also optimal for designing DMT with orthonormal transmitter [5].

In this paper, we will formulate the design problem of optimal DMT systems and point out the duality in the design of optimal DMT systems and optimal biorthogonal filter banks. We will show that the design of optimal perfect DMT systems can be converted to the design problem of a hypothetical subband coder and hence can be solved using existing techniques for designing optimal biorthogonal filter banks in most cases.

## 2. THE DMT SYSTEM

In this section, we will formulate the problem of designing optimal DMT system for a given channel $C(z)$ and channel

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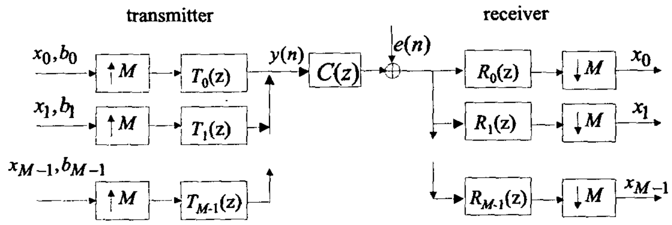


Figure 1: An $M$-band DMT system over a frequency selective channel $C(z)$ with additive channel noise $e(n)$.

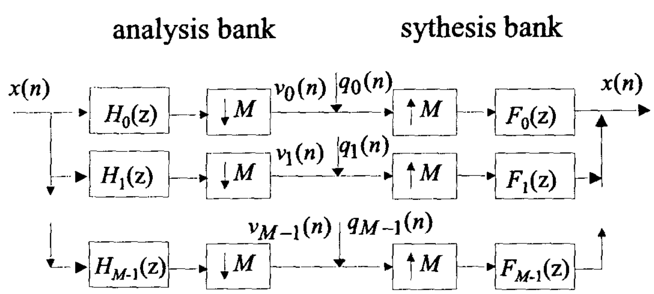


Figure 2: An $M$-band subband coder.

noise $e(n)$. For simplicity, we assume $x_k(n)$ are PAM symbols. Each of the symbols $x_k(n)$ of the $k$-th band carries $b_k$ bits. The average bit rate is $b = 1/M \sum_{k=0}^{M-1} b_k$. In this case, the probability of symbol error of the $k$-band is related to $b_k$, the variance of $k$-th band symbols $\sigma^2_{x_k}$ and the $k$-th output noise variance $\sigma^2_{e_k}$ by

$$P_e(k) = 2(1-2^{-b_k})Q\left(\sqrt{\frac{3\sigma^2_{x_k}}{(2^{2b_k}-1)\sigma^2_{e_k}}}\right),$$

where $Q(y) = \frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-t^2/2}dt$, for $y \geq 0$. Given a fixed probability of error $P_e$ for all bands, we have

$$\sigma^2_{x_k} = g(P_e, b_k)\sigma^2_{e_k}, \tag{1}$$

$$\text{where } g(P_e, b_k) = \left[Q^{-1}\left(\frac{P_e}{2(1-2^{-b_k})}\right)\right]^2 \frac{2^{2b_k}-1}{3}.$$

***Transmitted power.*** Assuming the input modulation symbols $x_k(n)$ are white and uncorrelated, which can always be done with proper interleaving. The transmitted power $\mathcal{P}$ is given by

$$\mathcal{P} = \sum_{k=0}^{M-1} \|t_k\|_2^2 \sigma^2_{x_k}, \tag{2}$$

where $\|t_k\|_2^2 = \sum_n |t_k(n)|^2$ is the energy of the $k$-th transmitting filter. On the other hand, the output noise of the $k$-th band is $\sigma^2_{e_k} = \int_0^{2\pi} S_{ee}(e^{j\omega})|R_k(e^{j\omega})|^2 \frac{d\omega}{2\pi}$, where $S_{ee}(e^{j\omega})$ is the power spectrum of the channel noise $e(n)$. Using (1) and the above expression for $\sigma^2_{e_k}$, we obtain the following expression for the transmitted power

$$\mathcal{P} = \sum_{k=0}^{M-1} g(P_e, b_k)\|t_k\|_2^2 \int_0^{2\pi} S_{ee}(e^{j\omega})|R_k(e^{j\omega})|^2 \frac{d\omega}{2\pi} \tag{3}$$

## 3. OPTIMAL BIORTHOGONAL SUBBAND CODERS

An $M$-band filter bank $\mathcal{F} = \{H_k(z), F_k(z)\}$ is as shown in Fig. 2. The quantization noises $q_k(n)$ are usually assumed to be wide sense stationary random processes that are white, zero mean and uncorrelated. The variance of the output quantization noise $q(n)$ is given by

$$\sigma_q^2 = \sum_{k=0}^{M-1} \|f_k\|_2^2 \sigma^2_{q_k},$$

where $\|f_k\|_2^2 = \sum_n |f_k(n)|^2$ is the energy of the $k$-th synthesis filter. The variance of the $k$-th quantization noise $q_k(n)$ is related to the variance of the $k$-th subband signal $v_k(n)$ by a distortion function,

$$\sigma^2_{q_k} = D(b_k)\sigma^2_{v_k},$$

where $b_k$ is the number of bits allocated to the $k$-th subband. An example of the distortion function is $D(b_k) = c2^{-2b_k}$ in the high bit rate case. The variance of the $k$-th subband signal $\sigma^2_{v_k}$ is given by $\sigma^2_{v_k} = \int_0^{2\pi} S_{xx}(e^{j\omega})|H_k(e^{j\omega})|^2 \frac{d\omega}{2\pi}$, where $S_{xx}(e^{j\omega})$ is the input power spectrum. Therefore,

$$\sigma_q^2 = \sum_{k=0}^{M-1} D(b_k)\|f_k\|_2^2 \int_0^{2\pi} S_{xx}(e^{j\omega})|H_k(e^{j\omega})|^2 \frac{d\omega}{2\pi} \tag{4}$$

***Principle Component Filter Banks (PCFB).*** In recent years, great advance has been made in the study of optimal orthonormal filter banks or the so-called Principle Component Filter Banks (PCFB) [6][8][9], . The development is based on the majorization theorem. Given 2 ordered sequences $\{a_n\}_{n=0}^{M-1}$ and $\{b_n\}_{n=0}^{M-1}$ with $a_n \geq a_{n+1}$ and $b_n \geq b_{n+1}$, we say $\{a_n\}_{n=0}^{M-1}$ majorizes $\{b_n\}_{n=0}^{M-1}$ if

$$\sum_{n=0}^{N} a_n \geq \sum_{n=0}^{N} b_n, \; 0 \leq N \leq M-1,$$

with equality when $N = M - 1$.

Consider a class of filter banks $\mathcal{C}$. The class can be the collection of FIR filter banks or the set of ideal filter banks. A filter bank $\mathcal{F}$ in the class $\mathcal{C}$ is a PCFB for the given input $S_{xx}(e^{j\omega})$ if the set $\{\sigma_{v_k}^2\}_{k=0}^{M-1}$ formed by its subband variances majorizes the set $\{\sigma_{v_k}'^2\}_{k=0}^{M-1}$ formed by the subband variances of any other filter bank $\mathcal{F}'$ in the class $\mathcal{C}$. The PCFB, when it exists, minimizes the output quantization noise in (4). This result does not require that $q_k(n)$ be white and uncorrelated. Also the PCFB is optimal for any given bit allocation. In particular, it is optimal under optimal bit allocation.

***Prefilters for Orthonormal Filter Banks.*** To minimize the quantization noise or to maximize the coding gain, [11] considers a class of biorthogonal filter banks by cascading orthonormal or paraunitary (PU) filter banks with pre- and post filters (Fig. 3). This will be called PPU structure. The

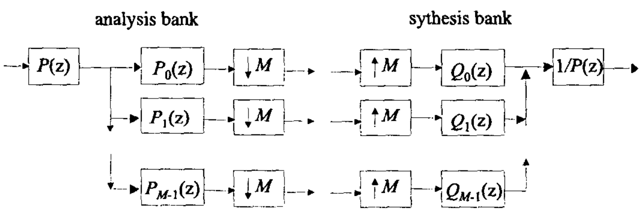


Figure 3: An $M$-band filter bank with pre-filter $P(z)$ and post filter $1/P(z)$.

analysis and synthesis filters of the biorthogonal filter banks are of the form

$$H_k(e^{j\omega}) = P(e^{j\omega})P_k(e^{j\omega}), \quad F_k(z) = \frac{P_k^*(e^{j\omega})}{P(e^{j\omega})}, \quad (5)$$

where $\{P_k(e^{j\omega}), P_k^*(e^{j\omega})\}$ form an orthonormal filter bank. Under high bit rate assumption $D(b_k) = c2^{-2b_k}$ and optimal bit allocation, it is shown that [11] the optimal prefilter $P(e^{j\omega})$ should be the half whitening filter for the input power spectrum $S_{xx}(e^{j\omega})$, i.e.,

$$P(e^{j\omega}) = 1/S_{xx}^{1/4}(e^{j\omega}).$$

Furthermore, $\{P_k(e^{j\omega}), P_k^*(e^{j\omega})\}$ should be the PCFB for the input power spectrum $\sqrt{S_{xx}(e^{j\omega})}$. That is, the design problem decouples as 2 problems: the problem of designing a half whitening filter for $S_{xx}(e^{j\omega})$ and the problem of designing the PCFB for $\sqrt{S_{xx}(e^{j\omega})}$.

***Optimal Biorthogonal Filter Banks.*** More recently, Moulin et. al. [10] shows that it is not a loss of generality assuming the PPU structure in designing optimal biorthogonal subband coders. It is shown in [10] that, for the objective function in (4), we can consider filter banks that is the cascade of an orthonormal (paraunitary) filter bank with pre-filter $P(e^{j\omega})$ and post filter $1/P(e^{j\omega})$ as in Fig. 3. The problem of designing optimal biorthogonal filter banks can be decoupled as the problem of designing a half whitening filter and a PCFB. Without assuming optimal bit allocation, this is true in most cases [10]. Only in some pathological cases, the function $D(\cdot)$ is subject to the following conditions: $D(\cdot)$ is strictly positive, strictly convex, and $\ln(D(\cdot))$ is concave. The high bit rate model $D(b_k) = c2^{-2b_k}$ is an example that satisfies the above assumptions.

## 4. OPTIMAL BIORTHOGONAL DMT SYSTEM

It is known that, when the channel $C(z)$ is ideal we can obtain a perfect $M$-band DMT from an $M$-band filter bank by using the synthesis bank and analysis bank of an $M$-band PR filter bank as the transmitter and receiver. In particular, given a PR filter bank $\mathcal{F} = \{H_k(z), F_k(z)\}$ the DMT system $\mathcal{D} = \{F_k(z), H_k(z)\}$ with transmitting filters $F_k(z)$ and receiving filters $H_k(z)$ is perfect. For frequency selective channels, the connection can be made more general as follows:

**Theorem 1** *A filter bank $\mathcal{F} = \{H_k(z), F_k(z)\}$ is biorthogonal if and only if the DMT system*

$$\mathcal{D} = \{F_k(z), H_k(z)/C(z)\}$$

*is perfect over a possibly non ideal channel $C(z)$.*

For every PR filter bank $\mathcal{F} = \{H_k(z), F_k(z)\}$, there is an associated perfect DMT $\mathcal{D} = \{F_k(z), H_k(z)/C(z)\}$.

***Orthonormal Transmitter.*** In an orthonormal filter bank with synthesis filters $F_k(e^{j\omega})$, the analysis filters are simply $F_k^*(e^{j\omega})$. Consider the DMT system with transmitting filters $T_k(z) = F_k(z)$, where $F_k(z)$ are the synthesis filters of an orthonormal filter bank $\{F_k^*(z), F_k(z)\}$. The receiving filters, by Theorem 1, are given by $F^*(e^{j\omega})/C(e^{j\omega})$. For the class of DMT system with orthonormal transmitters, the design problem can be converted to the problem of designing an optimal orthonormal filter bank for an appropriately defined power spectrum [4]. More recently, it has been shown that the design of the optimal DMT system of this class can be further formulated as a PCFB design problem. The optimal DMT system with orthonormal transmitter can be obtained by choosing the associated orthonormal filter bank $\mathcal{F} = \{F_k^*(z), F_k(z)\}$ to be the PCFB for the input power spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$. In the following we consider the more general DMT systems, where the associated PR filter banks are biorthogonal, not restricted to the orthonormal case.

***DMT with PPU structure.*** Here we consider the case that the associated PR filter bank of the DMT system is of the PPU structure in Sec. 3. To be more specific, the transmitting filters $T_k(z)$ and receiving filters $R_k(z)$ are given by,

$$T_k(z) = \frac{P_k(e^{j\omega})}{P(e^{j\omega})}, R_k(e^{j\omega}) = \frac{P_k^*(e^{j\omega})}{C(e^{j\omega})} P(e^{j\omega}), \qquad (6)$$

where $\{P_k(e^{j\omega}), P_k^*(e^{j\omega})\}$ forms an orthonormal filter bank. It can be shown that [12] the problem of designing optimal DMT of this class (referred to as the DMT systems of the PPU class) also decouples as in the design of optimal subband coders with PPU structure. This result is summarized in the following theorem.

**Theorem 2** *Consider the DMT system of the PPU class with transmitting and receiving filters as given in (6). The optimal prefilter $P(e^{j\omega})$ is a half whitening filter for the power spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$. That is,*

$$P(e^{j\omega}) = \frac{1}{(S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2)^{1/4}}.$$

*The associated orthonormal filter bank $\{P_k(e^{j\omega}), P_k^*(e^{j\omega})\}$ should be the PCFB for the power spectrum $\left(\frac{S_{ee}(e^{j\omega})}{|C(e^{j\omega})|^2}\right)^{1/2}$.*

This result holds for any given bit allocation. Note that, in the design of optimal filter banks with a PPU structure, we need to assume that the quantization noise $q_k(n)$ are white and uncorrelated. In the DMT design problem, we can always perform appropriate interleaving so that the input modulation symbols $x_k(n)$ are white and uncorrelated.

***Optimal DMT.*** For the design of the optimal DMT systems for the most general class, let us consider the DMT system

$$\mathcal{D} = \{F_k(z), H_k(z)/C(z)\},$$

where $H_k(z)$ and $F_k(z)$ are the analysis and synthesis filters of a biorthogonal filter bank $\mathcal{F} = \{H_k(z), F_k(z)\}$. By Theorem 1, we know there is no loss of generality in such a construction. The transmitted power in (3) can be rewritten as

$$\mathcal{P} = \sum_{k=0}^{M-1} g(P_e, b_k)\|f_k\|^2 \int_0^{2\pi} \frac{S_{ee}(e^{j\omega})}{|C(e^{j\omega})|^2} |H_k(e^{j\omega})|^2 \frac{d\omega}{2\pi} \qquad (7)$$

For the above objective function, we can convert it to the following hypothetical filter bank design problem: consider the $M$-band filter bank in Fig. 2 with input power spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$. Suppose the distortion function $D(b_k)$ is replaced by $g(P_e, b_k)$; the variance of quantization noise $q_k(n)$ is

$$\sigma_{q_k}^2 = g(P_e, b_k)\sigma_{v_k}^2.$$

Then the output quantization noise $\sigma_q^2$ is given exactly by (7)! Note that in the design of optimal biorthogonal filter banks, the problem decouples as prefilter design and PCFB design in most cases without making assumptions on $D(b_k)$. This means that, except in pathological cases, we can solve the design problem of optimal perfect DMT systems in the same manner using the design method for optimal biorthogonal filter banks .

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