AoA Estimation Based on Fourth-Order Coarrays and Cumulants for Mmwave Systems

Yuan-Pei Lin and Ting-Ming Yang

***Abstract*—In radar applications, it is known that with a sparse array of N antennas, we can construct a virtual array of size $O(N^4)$ using fourth-order coarrays and cumulants. Such a result is valid when the sources are not Gaussian. In this paper, we consider the estimation of angles of arrival (AoA) for mmWave channels using fourth-order coarrays. An mmWave system usually has a large number of antennas but few RF chains, which restricts the effective antenna size. The angle estimation problems of mmWave and radar systems are similar, but there are differences. In particular, in the problem formulation of angle estimation for mmWave systems, we have path gains instead of source symbols. The path gains in mmWave channels are typically modeled as Gaussian random variables. We show that the critical non-Gaussian assumption for using fourth-order coarrays can be circumvented by using a submatrix of a banded Toeplitz matrix for analog combining. The combiner can be designed so that the conditions for applying fourth-order coarrays are satisfied asymptotically. A virtual array of size $O(N^4)$ can be constructed when there are N RF chains.**

***Index Terms*—Channel estimation, higher order statistics, massive MIMO.**

I. Introduction

IN THE literature, direction finding for radar applications has been extensively studied. With a uniform linear array (ULA) with N antennas, up to $N-1$ sources can be identified using subspace-based methods [1], [2], [3]. The number of identifiable sources can be increased significantly by using a sparse array [4], [5], [6] based on second-order coarrays and second-order statistics. Through judicious placement of the antennas, an enlarged ULA of size $O(N^2)$ can be constructed. It is possible to identify up to $O(N^2)$ sources using N sensors [6]. On the other hand, direction finding based on fourth-order cumulants is known to be more robust to Gaussian noise as the fourth-order cumulant of Gaussian noise is zero [7], [8]. Extension of sparse arrays based on fourth-order cumulants instead of second-order statistics is proposed in [9]. Using fourth-order coarrays, it is possible to construct a virtual ULA array of size $O(N^4)$ based on fourth-order cumulants [10], [11]. The result holds under the assumption that the sources are not Gaussian.

Recent advances show that for mmWave systems, it is feasible to pack a large number of antennas in a small area due to small wavelengths. In spite of a large array size, cost and power constraints often limit the number of radio frequency (RF) chains that perform downconversion and sampling [12], [13]. The RF chain limitation restricts the effective array size to the number of RF chains N. Thus it is of great interest to have augmented virtual arrays for mmWave systems. It is shown in [14] when the combiner is a Submatrix of Banded Toeplitz matrices (SBT), we can use second-order coarrays and second-order statistics to have virtual arrays of size $O(N^2)$. SBT matrices, called convolutional beamspace transformation in [15], have also been proposed for low-complexity angle estimation in radar applications when each antenna has a designated RF chain.

In this paper we consider the estimation of AoA for mmWave channels using fourth-order coarrays based on cumulants. In the problem formulation of AoA estimation for mmWave systems, the path gains of the channel take the place of sources symbols in radar applications. The path gains are typically modeled as Gaussian random variables and thus the critical non-Gaussian assumption for using fourth-order coarrays is not satisfied. We show that the non-Gaussian assumption can be circumvented using an SBT combiner. The combiner can be designed so that the conditions for applying fourth-order coarrays can be satisfied asymptotically. Given N RF chains, we can obtain a virtual array of size $O(N^4)$. Although the results are derived based on fourth-order statistics, simulations are given to demonstrate that the proposed method can be used even for a moderate number of training vectors.

Notation: The variance of a random variable x is denoted as σ_x^2 and the expectation of x by $E[x]$. The ℓ-th entry of a vector $\mathbf{x}$ is denoted as $[\mathbf{x}]_\ell$. The notation $\mathbf{A}^T$, $\mathbf{A}^*$, and $\mathbf{A}^H$ denote, respectively, the transpose, the conjugate, and conjugate transpose of a matrix $\mathbf{A}$. $\mathbf{I}_n$ denotes an $n \times n$ identity matrix. $diag(\boldsymbol{\lambda})$ denotes a diagonal matrix whose diagonal entries are those of a vector $\boldsymbol{\lambda}$. Given an $m \times n$ matrix $\mathbf{X}$, we use $vec(\mathbf{X})$ to denote the $mn \times 1$ vectorized version of $\mathbf{X}$. The notation $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A} \odot \mathbf{B}$ denote, respectively, the Kronecker product and the Khatri-Rao product. A useful property concerning the Khatri-Rao product is as follows. Let $\mathbf{X} = \mathbf{A}\text{diag}(\boldsymbol{\lambda})\mathbf{B}$, then $vec(\mathbf{X}) = (\mathbf{B}^T \odot \mathbf{A})\boldsymbol{\lambda}$.

II. Review of Angle Estimation Using Fourth-Order Coarrays and Cumulants

We first give definitions of fourth-order cumulants and coarrays. Then we present a brief review on direction finding using coarrays based on fourth-order cumulants.

Fourth-order cumulants [7]: Given an $N \times 1$ vector $\mathbf{y}$, the fourth-order cumulant matrix of $\mathbf{y}$ is the $N^2 \times N^2$ matrix

$$\mathbf{C}_y = E[(\mathbf{y} \otimes \mathbf{y}^*)(\mathbf{y} \otimes \mathbf{y}^*)^H] - E[\mathbf{y} \otimes \mathbf{y}^*]E[(\mathbf{y} \otimes \mathbf{y}^*)^H]$$

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$$- E[\mathbf{y}\mathbf{y}^H] \otimes E[(\mathbf{y}\mathbf{y}^H)^*]. \quad (1)$$

When y is a scalar, $C_y = E[|y|^4] - 2E[|y|^2]$. In the case y is Gaussian, $C_y = 0$.

Fourth-order coarrays [9]: Let $\mathcal{S}$ be the set of integers $\{n_0, n_1, \ldots, n_{N-1}\}$. Define the fourth-order difference coarray of $\mathcal{S}$ as $\mathcal{D} = \{n_i - n_j - n_\ell + n_m, 0 \leq i, j, \ell, m < N\}$. Suppose $\mathcal{D}$ contains consecutive integers $-P, -P+1, \ldots, P$. Then $\mathcal{D}$ is said to have *degree of freedom (DoF)* equal to $2P + 1$. For example, let $N = 3$ and $\mathcal{S} = \{0, 1, 4\}$, then $\mathcal{D} = \{n : -8 \leq n \leq 8\}$. As $\mathcal{D}$ contains 17 consecutive integers, from -8 to 8, we have $P = 8$ and DoF $= 17$.

Consider N antennas that are placed according to $\mathcal{S} = \{n_0, n_1, \ldots, n_{N-1}\}$, i.e., located at $n_0 d, n_1 d, \ldots, n_{N-1} d$, where d is the smallest distance between any two antennas. Suppose there are L sources impinging on the array with AoAs $\theta_1, \theta_2, \ldots, \theta_L$. The received vector is

$$\mathbf{y}_k = \mathbf{A}\mathbf{x}_k + \mathbf{n}_k, \quad (2)$$

where $\mathbf{x}_k$ is the $L \times 1$ source vector, and $\mathbf{A}$ is the antenna response matrix. The ℓ-th column of $\mathbf{A}$ is given by $\mathbf{a}_\ell = [e^{jn_0 u_\ell} \quad e^{jn_1 u_\ell} \quad \cdots \quad e^{jn_{N-1} u_\ell}]^T$, where $u_\ell = (2\pi d/\lambda) \sin \theta_\ell$, and λ is the wavelength. Let $\mathbf{C}_x$, $\mathbf{C}_s$ and $\mathbf{C}_n$ be, respectively, the fourth-order cumulant matrices of the source vector $\mathbf{x}_k$, the signal part $\mathbf{A}\mathbf{x}_k$ and the noise $\mathbf{n}_k$ in (2). Suppose the following conditions are satisfied:

1) The sources are independent non-Gaussian random variables that have zero mean.
2) The noise $\mathbf{n}_k$ is circularly symmetric Gaussian with zero mean and $E[\mathbf{n}_k \mathbf{n}_k^H] = \sigma_n^2 \mathbf{I}_N$ so that $\mathbf{C}_n = \mathbf{0}$.
3) The noise and the sources are independent so that $\mathbf{C}_y = \mathbf{C}_s + \mathbf{C}_n$

When Conditions (1)–(3) hold, $\mathbf{C}_y$ can be nicely expressed as [9]

$$\mathbf{C}_y = (\mathbf{A} \odot \mathbf{A}^*) \text{diag}(\boldsymbol{\lambda}_x) (\mathbf{A} \odot \mathbf{A}^*)^H, \quad (3)$$

where $\boldsymbol{\lambda}_x$ is $L \times 1$ and $[\boldsymbol{\lambda}_x]_\ell$ is equal to the fourth-order cumulant of $[\mathbf{x}_k]_\ell$. For Gaussian sources, $\boldsymbol{\lambda}_x = \mathbf{0}$ and thus the sources should be non-Gaussian (Condition (1)). The vectorized version of $\mathbf{C}_y$ is given by $\mathbf{c}_y = \mathbf{B}\boldsymbol{\lambda}_x$, where $\mathbf{B} = \mathbf{A}^* \odot \mathbf{A} \odot \mathbf{A} \odot \mathbf{A}^*$. It turns out that $\mathbf{B}$ contains an antenna response matrix corresponding to an enlarged ULA of size $2P + 1$. We can extract rows of $\mathbf{B}$ to form a ULA matrix. In particular, we can find a selection matrix $\mathbf{J}$ such that $\tilde{\mathbf{A}} = \mathbf{J}\mathbf{B}$ is the antenna response matrix corresponding to a ULA of $(2P + 1)$ antennas. The ℓ-th column vector of $\tilde{\mathbf{A}}$ is given by $[e^{-jPu_\ell} \quad e^{-j(P-1)u_\ell} \quad \cdots e^{jPu_\ell}]$. Then $\tilde{\mathbf{c}}_y = \mathbf{J}\mathbf{c}_y$ is equal to $\tilde{\mathbf{c}}_y = \tilde{\mathbf{A}}\boldsymbol{\lambda}_x$. Define a $(P+1) \times (P+1)$ square matrix $\tilde{\mathbf{C}}_y$ whose n-th column is $\tilde{\mathbf{c}}_y[P - n : 2P - n]$, i.e, the $(P - n)$-th to the $(2P - n)$-th elements of $\tilde{\mathbf{c}}_y$, for $n = 0, 1, \ldots, P$. We have [9], $\tilde{\mathbf{C}}_y = \mathbf{A}_{P+1} \text{diag}(\boldsymbol{\lambda}_x) \mathbf{A}_{P+1}^H$, where $\mathbf{A}_{P+1}$ is a ULA antenna response matrix with the ℓ-th column given by $[1 \quad e^{ju_\ell} \quad \cdots \quad e^{jPu_\ell}]^T$ The matrix $\tilde{\mathbf{C}}_y$ is the same as an autocorrelation matrix corresponding to the output of a ULA of size $P + 1$. This means, by placing the antennas according to $\mathcal{S}$ and using fourth-order cumulants and coarrays, we obtain a virtual ULA array of size $P + 1$. The estimation of AoA for a $(P + 1)$-sensor ULA array can be solved using many methods [3]. When subspace based methods like MUSIC or ESPRIT are used, up to P sources can be identified. Various approaches have been proposed to design the set $\mathcal{S}$ so that P is in the order of N^4 [10], [11].

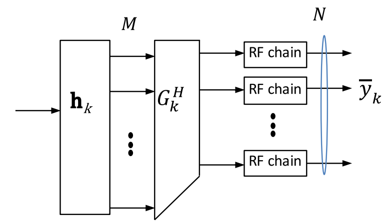

Fig. 1. An mmWave system with a single transmit antenna and a receiver equipped with M antennas and N RF chains.

III. Angle Estimation for Mmwave Systems Using Fourth-Order Coarrays

Consider the mmWave system with M receive antennas and a single transmit antenna in Fig. 1. When there are L paths, the channel at time k is an $M \times 1$ vector $\mathbf{h}_k$ of the form

$$\mathbf{h}_k = \mathbf{A}_M \boldsymbol{\alpha}_k,$$

where $\mathbf{A}_M$ is the $M \times L$ antenna response matrix. When ULA is used, the ℓ-th column of $\mathbf{A}_M$ is given by $[1 \quad e^{ju_\ell} \quad \cdots \quad e^{j(M-1)u_\ell}]^T$ and u_ℓ is as in (2). The $L \times 1$ vector $\boldsymbol{\alpha}_k$ consists of complex path gains of the L paths at times k. The path gains are typically modeled as independent Gaussian random variables [13]. Suppose the receiver is equipped with N RF chains, where $M >> N$. The RF combiner $\mathbf{G}_k$ in Fig. 1 at time k is of size $M \times N$. The output vector of the combiner at time k is [16]

$$\bar{\mathbf{y}}_k = \sqrt{P_t} \mathbf{G}_k^H \mathbf{A}_M \boldsymbol{\alpha}_k + \mathbf{G}_k^H \mathbf{n}_k, \quad (4)$$

where the channel noise $\mathbf{n}_k$ is as in Condition (2) and P_t is the transmit power. Notice that if the combiner $\mathbf{G}_k$ is an antenna selection matrix that chooses the n_0-th, n_1-th,..., and n_{N-1}-th antennas, then the received vector in (4) is exactly the same as that in (2). In this case the path gains need to be non-Gaussians for using fourth-order coarrays. In the following, we show how to take advantage of the combiner so that we can use fourth-order coarrays when the path gains are Gaussian.

Suppose the combiner is a Submatrix of an $M \times M$ lower triangular Banded Toeplitz matrix (SBT). It is obtained by choosing N columns according to a set $\mathcal{S} = \{n_0, n_1, \ldots, n_{N-1}\}$. For example, let $N = 3$, and $\mathcal{S} = \{0, 1, 4\}$, then we choose column numbers $0, 1, 4$ and the combiner is of the form

$$\mathbf{G}_k^H = \begin{bmatrix} \boxed{\quad \mathbf{g}_k^H \quad} & 0 & 0 & 0 & 0 \\ 0 & \boxed{\quad \mathbf{g}_k^H \quad} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\quad \mathbf{g}_k^H \quad} \end{bmatrix}.$$

The vector $\mathbf{g}_k$ that contains the nonzero coefficients is referred to as the prototype of $\mathbf{G}_k$. Define the discrete time Fourier transform of $\mathbf{g}_k$ as $G_k(\omega) = \sum_{m=0}^{M-1} [\mathbf{g}_k]_m e^{-jm\omega}$. The following result can be obtained from [14], [15].

Lemma 1 [14], [15]: Suppose the combiner $\mathbf{G}_k$ is SBT, then the received vector $\bar{\mathbf{y}}_k$ in (4) is given by

$$\bar{\mathbf{y}}_k = \mathbf{A}\bar{\boldsymbol{\alpha}}_k + \bar{\mathbf{n}}_k, \quad (5)$$

where $\mathbf{A}$ is the same as the antenna response matrix in (2), $\bar{\boldsymbol{\alpha}}_k$ is the equivalent path gain vector with $[\bar{\boldsymbol{\alpha}}_k]_\ell = \sqrt{P_t}G_k(u_\ell)[\boldsymbol{\alpha}_k]_\ell$ and $\bar{\mathbf{n}}_k = \mathbf{G}_k^H \mathbf{n}_k$ is the equivalent noise.

Lemma 1 means that, with SBT combining, we have a virtual linear array with elements located at $n_0\, d, n_1\, d, \ldots, n_{N-1} d$ even though $\mathbf{G}_k$ is not an antenna selection matrix. The received signal in (5) is of the form in (2), but the equivalent path gains and noise are now partially controlled by the combiner of our design. We can design the combiner so that Conditions (1)–(3) are satisfied and fourth-order coarray results can be applied. Suppose the prototype $\mathbf{g}_k$ consists of independent binary random variables, i.e.,

$$[\mathbf{g}_k]_m = -\frac{1}{\rho} \text{ or } \frac{1}{\rho} \quad \text{with equal probability,} \tag{6}$$

where $\rho = \sqrt{M - n_{N-1}}$ so that $\mathbf{g}_k^H \mathbf{g}_k = 1$.

Proposition 1: When the prototype of the combiner $\mathbf{G}_k$ is chosen as in (6), the equivalent noise $\bar{\mathbf{n}}_k$ is zero-mean Gaussian with $E[\bar{\mathbf{n}}_k \bar{\mathbf{n}}_k^H] = \sigma_n^2 \mathbf{I}_N$ and $\mathbf{C}_{\bar{n}} = \mathbf{0}$.

See Appendix A for a proof. Proposition 1 implies that the equivalent noise $\bar{\mathbf{n}}_k$ is the same as the actual channel noise statistically and it satisfies Condition (2).

Proposition 2: Consider the prototype $\mathbf{g}_k$ in (6) with a fixed number of RF chains. Then $G_k(u_1), G_k(u_2), \ldots G_k(u_L)$ are asymptotically independent Gaussian random variables of zero mean and unit variance as M goes to infinity.

See Appendix B for a proof. Using Proposition 2 we can conclude that if $[\boldsymbol{\alpha}_k]_\ell$'s are independent and Gaussian, the equivalent path gains $[\bar{\boldsymbol{\alpha}}_k]_\ell = \sqrt{P_t}G_k(u_\ell)[\boldsymbol{\alpha}_k]_\ell$ are asymptotically independent double Gaussian [17] and hence non-Gaussian, as required in Condition (1). As both the equivalent path gain $\bar{\boldsymbol{\alpha}}_k$ and noise $\bar{\mathbf{n}}_k$ depend on the combiner, they are not independent. We can not use the independence of the two to obtain the simple expression $\mathbf{C}_{\bar{y}} = \mathbf{C}_{\bar{s}} + \mathbf{C}_{\bar{n}}$ with no cross terms like (2). Nonetheless, we have the desired result when M is large.

Proposition 3: Consider the prototype $\mathbf{g}_k$ in (6) and a fixed number of RF chains. When the number of antennas M goes to infinity, we have $\mathbf{C}_{\bar{y}} = \mathbf{C}_{\bar{s}} + \mathbf{C}_{\bar{n}}$, where $\mathbf{C}_{\bar{s}}$ and $\mathbf{C}_{\bar{n}}$ are, respectively, the fourth-order cumulant matrices of the signal part $\mathbf{A}\bar{\boldsymbol{\alpha}}_k$ and the noise $\bar{\mathbf{n}}_k$.

See Appendix C for a proof. Propositions 1–3 imply Conditions (1)–(3) are met and we have the following theorem.

Theorem 1: For the prototype $\mathbf{g}_k$ in (6), the fourth-order cumulant matrix of the received $\bar{\mathbf{y}}_k$ satisfies

$$\mathbf{C}_{\bar{y}} = \mathbf{C}_{\bar{s}} = (\mathbf{A} \odot \mathbf{A}^*)\text{diag}(\boldsymbol{\lambda}_{\bar{\alpha}})(\mathbf{A} \odot \mathbf{A}^*)^H, \tag{7}$$

as M goes to infinity.

As (7) is of the same form in (3), we can follow the steps in Section 2 and obtain a $(P+1) \times (P+1)$ hypothetical autocorrelation matrix

$$\tilde{\mathbf{C}}_{\bar{y}} = \mathbf{A}_{P+1}\text{diag}(\boldsymbol{\lambda}_{\bar{\alpha}})\mathbf{A}_{P+1}^H.$$

The acquisition of angles becomes a problem of AoA estimation for a $(P+1)$-sensor ULA as in Section 2 and up to P paths can be identified. Similar to the results in Section 2, we can choose the set $\mathcal{S}$ so that P is in the order of N^4. Note that the SBT combiner $\mathbf{G}_k$ in Fig. 1 contains nonzero and zero coefficients. With the prototype in (6), the nonzero coefficients are of unit magnitude and they can be implemented using phase shifters. The zero coefficients can be implemented using switches [20].

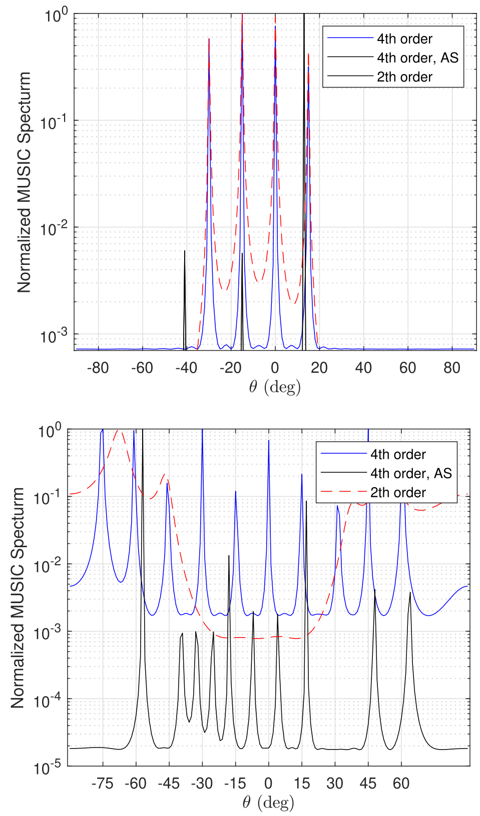

Fig. 2. Normalized MUSIC spectra; top: a four-path channel and bottom: a ten-path channel.

IV. Simulations

Consider an mmWave receiver with $M = 64$ antennas, $N = 4$ RF chains, and half wavelength antenna spacing. The combiner is SBT, designed according to $\mathcal{S} = \{0, 1, 9, 12\}$ and thus $\mathcal{D} = \{n : -24 \leq n \leq 24\}$. We choose the prototype of the combiner as in (6); we have a virtual ULA of size 25 and up to 24 paths can be identified. The path gains of the mmWave channel are Gaussian distributed with zero mean and unit variance. We examine the MUSIC spectra for different L. Using the received vector in (4), we compute the empirical fourth-order cumulant matrix $\hat{\mathbf{C}}_{\bar{y}}$. Let the singular value decomposition of $\hat{\mathbf{C}}_{\bar{y}}$ be $\mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^H$, where $\mathbf{U} = [\mathbf{U}_s \quad \mathbf{U}_n]$ with $\mathbf{U}_s$ corresponding to the signal subspace and $\mathbf{U}_n$ corresponding to the noise subspace. Fig. 2 shows the MUSIC spectrum $1/(\mathbf{a}^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\theta))$, where $\mathbf{a}(\theta) = [1 \quad e^{j\pi \sin\theta} \quad \cdots \quad e^{j\pi(M-1)\sin\theta}]^T$ for SNR $1/\sigma_n^2 = 10$ dB and 5,000 training vectors. In Fig. 2(a), the number of paths L is equal to 4, and $\theta_\ell = -45^\circ + 15^\circ\ell$, for $\ell = 1, 2, \ldots, 4$. In Fig. 2(b), $L = 10$ and $\theta_\ell = -90^\circ + 15^\circ\ell$, for $\ell = 1, 2, \ldots, 10$. We can see that all the paths can be resolved for $L = 4$ and $L = 10$ with the proposed combiner based on fourth-order coarrays. For comparison, we show the MUSIC spectrum when $\mathbf{G}_k$ is an antenna selection (AS) matrix that selects the antennas in $\mathcal{S} = \{0, 1, 9, 12\}$. In this case, the path gains remain Gaussian, the fourth-order cumulants of the path gains are zero, and the paths can not be identified. We have also shown the MUSIC spectrum when second-order coarrays and second-order statistics [14] are used. With second-order coarrays, at most 6 paths can be identified for $N = 4$ [18]. In this case four peaks are resolved in Fig. 2(a) as $L < 6$, but the paths can not be successfully identified in Fig. 2(b) as $L > 6$.

V. Conclusion

In this paper, we show that fourth-order coarrays can be used for angle estimation of mmWave systems with a limited number of RF chains. With SBT combining, the conditions for applying fourth-order coarrays can be satisfied asymptotically. Virtual arrays of augmented sizes can be constructed, much like radar applications. Simulations are given to show that using the proposed combiner and fourth-order coarrays more paths can be identified than the case without combining.

Appendix A
Proof of Proposition 1

Note that $[\bar{\mathbf{n}}_k]_i = \sum_{m=0}^{M-n_{N-1}} [\mathbf{g}_k]_m [\mathbf{n}_k]_{m+n_i}$, i.e., a sum of independent random variables. It can be verified that each term $[\mathbf{g}_k]_m [\mathbf{n}_k]_{m+n_i}$ is Gaussian, of zero mean and variance σ_n^2/ρ^2 when $[\mathbf{g}_k]_m$ is a binary random variable as in (6). As $[\bar{\mathbf{n}}_k]_i$ is a sum of ρ^2 Gaussian random variables that have zero mean and variance σ_n^2/ρ^2, it is Gaussian, of zero mean and variance σ_n^2. It remains to show that the elements of $\bar{\mathbf{n}}_k$ are independent. For the prototype in (6), we have $E[\mathbf{G}_k^H \mathbf{G}_k] = \mathbf{I}_N$, and thus $E[\bar{\mathbf{n}}_k \bar{\mathbf{n}}_k^H] = \sigma_n^2 \mathbf{I}_N$. This means the elements of $\bar{\mathbf{n}}_k$ are uncorrelated and hence independent as they are Gaussian.

Appendix B
Proof of Proposition 2

Proposition 2 can be proved by showing the following two results hold as M goes to infinity. (1) $G_k(\omega)$ converges asymptotically to a Gaussian random variable of zero mean and unit variance for all ω. (2) $G_k(\omega_1)$ and $G_k(\omega_2)$ are asymptotically independent whenever $\omega_1 \neq \omega_2$.

(1) Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables, where X_i is of mean μ_i and variance σ_i^2. Define $s_n^2 = \sum_{i=1}^n \sigma_i^2$. The Lyapunov central limit theorem [19] states that the sum $\frac{1}{s_n}\sum_{i=1}^n (X_i - \mu_i)$ converges in distribution to a Gaussian random variable of zero mean and unit variance if there exists $\epsilon > 0$ such that $\lim_{n\to\infty} \frac{1}{s_n^{2+\epsilon}} \sum_{i=1}^n E[|X_i - \mu_i|^{2+\epsilon}] = 0$. Let $f_{k,m}(\omega) = \rho[\mathbf{g}_k]_m e^{-jm\omega}$, then $f_{k,m}(\omega)$ is of zero mean and unit variance and $G_k(\omega) = \sum_{m=0}^{M-n_{N-1}-1} \frac{1}{\rho} f_{k,m}(\omega)$. By the Lyapunov central limit theorem, $G_k(\omega)$ converges to a unit-variance Gaussian random variable of zero mean if we can find $\epsilon > 0$ so that $\lim_{M\to\infty} \frac{1}{\rho^{2+\epsilon}} \sum_{m=0}^{M-n_{N-1}-1} E[|f_{k,m}(\omega)|^{2+\epsilon}] = 0$. As $|f_{k,m}(\omega)| = 1$, the above condition holds if we choose $\epsilon = 1$.

(2) It is shown in [14] that $|E[G_k(\omega_1)G_k^*(\omega_2)]| = \frac{1}{(M-n_{N-1})} \times \frac{|\sin(\frac{1}{2}(\omega_1-\omega_2)(M-n_{N-1}))|}{|\sin(\frac{1}{2}(\omega_1-\omega_2))|}$. This means when M goes to infinity $|E[G_k(\omega_1)G_k^*(\omega_2)]|$ approaches zero, i.e., $G_k(\omega_1)$ and $G_k(\omega_2)$ are uncorrelated, whenever $\omega_1 \neq \omega_2$. As $G_k(\omega_1)$ and $G_k(\omega_2)$ are Gaussian, uncorrelatedness implies independence.

Appendix C
Proof of Proposition 3

For brevity, we omit the time index k in this proof and express $\bar{\mathbf{y}}_k$ in (4) as $\bar{\mathbf{y}} = \sqrt{P_t}\mathbf{G}^H \mathbf{A}_M \boldsymbol{\alpha} + \mathbf{G}^H \mathbf{n} = \mathbf{A}\bar{\boldsymbol{\alpha}} + \bar{\mathbf{n}} = \bar{\mathbf{s}} + \bar{\mathbf{n}}$. Using (1), we get

$$\begin{aligned}
\mathbf{C}_{\bar{y}} &= \mathbf{C}_{\bar{s}} + \mathbf{C}_{\bar{n}} + \sum_{i=1}^{4} \mathbf{T}_i + \mathbf{T}_i^H, \\
\mathbf{T}_1 &= E[(\bar{\mathbf{s}} \otimes \bar{\mathbf{n}}^*)(\bar{\mathbf{n}} \otimes \bar{\mathbf{s}}^*)^H], \\
\mathbf{T}_2 &= E[(\bar{\mathbf{s}} \otimes \bar{\mathbf{s}}^*)(\bar{\mathbf{s}} \otimes \bar{\mathbf{n}}^*)^H] + E[(\bar{\mathbf{s}} \otimes \bar{\mathbf{s}}^*)(\bar{\mathbf{n}} \otimes \bar{\mathbf{s}}^*)^H] \\
&\quad + E[(\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}^*)(\bar{\mathbf{s}} \otimes \bar{\mathbf{n}}^*)^H] + E[(\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}^*)(\bar{\mathbf{n}} \otimes \bar{\mathbf{s}}^*)^H], \\
\mathbf{T}_3 &= E[(\bar{\mathbf{s}} \otimes \bar{\mathbf{s}}^*)(\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}^*)^H] - E[\bar{\mathbf{s}} \otimes \bar{\mathbf{s}}^*] E[\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}^*]^H, \\
\mathbf{T}_4 &= E[(\bar{\mathbf{s}}\bar{\mathbf{s}}^H) \otimes (\bar{\mathbf{n}}\bar{\mathbf{n}}^H)^*] - E[\bar{\mathbf{s}}\bar{\mathbf{s}}^H] \otimes E[\bar{\mathbf{n}}\bar{\mathbf{n}}^H]^*. \qquad (8)
\end{aligned}$$

We show below that the elements of $\mathbf{T}_i$ are either zero or approach zero as M goes to infinity.

For $\mathbf{T}_1$, we observe that $\bar{\mathbf{s}} \otimes \bar{\mathbf{n}}^* = \mathbf{P}(\bar{\mathbf{n}}^* \otimes \bar{\mathbf{s}})$ for some properly chosen permutation matrix $\mathbf{P}$. We have $\mathbf{T}_1 = \mathbf{P}E[(\bar{\mathbf{n}}^* \otimes \bar{\mathbf{s}})(\bar{\mathbf{n}} \otimes \bar{\mathbf{s}}^*)^H]$, which is equal to $\mathbf{P}E[(\bar{\mathbf{n}}\bar{\mathbf{n}}^T)^* \otimes (\bar{\mathbf{s}}\bar{\mathbf{s}}^T)^H]$, i.e., $\mathbf{P}E[(\mathbf{G}\mathbf{n}\mathbf{n}^T\mathbf{G}^T)^* \otimes (\bar{\mathbf{s}}\bar{\mathbf{s}}^T)^H]$. Notice that $E[\bar{\mathbf{n}}\bar{\mathbf{n}}^T] = \mathbf{0}$ as $\mathbf{n}$ is complex circularly symmetric Gaussian. Using $\bar{\mathbf{s}} = \sqrt{P_t}\mathbf{G}^H \mathbf{A}_M \boldsymbol{\alpha}$ and that $\mathbf{n}$, $\mathbf{G}$ and $\boldsymbol{\alpha}$ are independent, we get $\mathbf{T}_1 = \mathbf{0}$.

The matrix $\mathbf{T}_2$ contains four terms. The first term $= E[(\bar{\mathbf{s}}\bar{\mathbf{s}}^H) \otimes (\bar{\mathbf{s}}\mathbf{n}^H\mathbf{G})^*] = E[(\bar{\mathbf{s}}\bar{\mathbf{s}}^H) \otimes (\bar{\mathbf{s}}E[\mathbf{n}]^H\mathbf{G})^*]$, which is zero as $\mathbf{n}$, $\mathbf{G}$ and $\boldsymbol{\alpha}$ are independent, and $\mathbf{n}$ is of zero mean. We can show the remaining terms are zero in a similar manner.

Express $\mathbf{T}_3$ as $\mathbf{Z} - \mathbf{Q}$, where $\mathbf{Z} = E[(\bar{\mathbf{s}} \otimes \bar{\mathbf{s}}^*)(\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}^*)^H]$ and $\mathbf{Q} = E[\bar{\mathbf{s}} \otimes \bar{\mathbf{s}}^*]E[\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}^*]^H$. Let $t_{i,j,\ell,\nu} = [\mathbf{T}_3]_{Ni+j,N\ell+\nu}$ and define $z_{i,j,\ell,\nu}$ and $q_{i,j,\ell,\nu}$ in a similar manner for $0 \le i,j,\ell,\nu < N$. Notice that $z_{i,j,\ell,\nu}$ is equal to $E[\mathbf{d}_i^H \mathbf{U} \mathbf{d}_j \mathbf{d}_\nu^H \mathbf{n}\mathbf{n}^H \mathbf{d}_\ell]$, where $\mathbf{d}_i$ is the i-th column of $\mathbf{G}$ and $\mathbf{U} = P_t E[\mathbf{A}_M \boldsymbol{\alpha}\boldsymbol{\alpha}^H \mathbf{A}_M^H]$. Using $E[\mathbf{n}\mathbf{n}^H] = \sigma_n^2 \mathbf{I}_N$, we get $z_{i,j,\ell,\nu} = \sigma_n^2 E[\mathbf{d}_i^H \mathbf{U}\mathbf{d}_j \mathbf{d}_\nu^H \mathbf{d}_\ell]$. On the other hand, $q_{i,j,\ell,\nu} = E[[\bar{\mathbf{s}}]_i[\bar{\mathbf{s}}^*]_j]E[[\bar{\mathbf{n}}^*]_\ell[\bar{\mathbf{n}}]_\nu]$. From Proposition 1, we know $q_{i,j,\ell,\nu} = \sigma_n^2 E[[\bar{\mathbf{s}}]_i[\bar{\mathbf{s}}^*]_j]$ for $\ell = \nu$ and it is zero otherwise. When $\ell = \nu$, we have $z_{i,j,\ell,\nu} = q_{i,j,\ell,\nu}$ and thus $[\mathbf{T}_3]_{Ni+j,N\ell+\nu} = 0$. It remains to show $t_{i,j,\ell,\nu} = 0$ for $\ell \neq \nu$. Let $\beta_p = \rho[\mathbf{G}]_{p,0}$, for $0 \le p < M - n_{N-1}$ and it is zero otherwise. Then $[\mathbf{d}_i]_p = \frac{1}{\rho}\beta_{p-n_i}$ and we have

$$t_{i,j,\ell,\nu} = \frac{1}{\rho^4} \sum_{r=0}^{M-1} \sum_{w=0}^{M-1} \sum_{p=0}^{M-1} u_{rw} \mu_{i,j,\ell,\nu}(r,w,p), \qquad (9)$$

where $\mu_{i,j,\ell,\nu}(r,w,p) = E[\beta_{r-n_i}\beta_{w-n_j}\beta_{p-n_\ell}\beta_{p-n_\nu}]$ and $u_{rw} = [\mathbf{U}]_{r,w}$. As β_p are independent, for $\ell \neq \nu$, we have $\mu_{i,j,\ell,\nu}(r,w,p) \neq 0$ only if $p - n_\nu = r - n_i, w - n_j$. Given r and w, the summation in (9) over p contains at most two nonzero terms, i.e., $p = r - n_i + n_\nu, w - n_j + n_\nu$. Similarly, as β_p are independent, each summation over w has at most one nonzero term and $|t_{i,j,\ell,\nu}| \le \frac{1}{\rho^4}\sum_{r=0}^{M-1} |u_{r,r-n_i+n_j-n_\ell+n_\nu}| + |u_{r,r-n_i+n_j+n_\ell-n_\nu}|$. Notice that $\mathbf{U} = P_t \mathbf{A}_M \mathbf{R}_\alpha \mathbf{A}_M^H$, where $\mathbf{R}_\alpha = E[\boldsymbol{\alpha}\boldsymbol{\alpha}^H]$. Let $r_\alpha = max_{i,j}|[\mathbf{R}_\alpha]_{i,j}|$. It follows that $max_{i,j}|[\mathbf{U}]_{i,j}| \le P_t L^2 r_\alpha$. Thus $|t_{i,j,\ell,\nu}| \le \frac{2}{(M-n_{N-1})^2}\sum_{r=0}^{M-1} 2P_t L^2 r_\alpha$, which approaches zero as M goes to infinity.

For $\mathbf{T}_4$, we observe that there is a one-to-one correspondence between the elements of $\mathbf{T}_3$ and $\mathbf{T}_4$. In particular, $[\mathbf{T}_4]_{Nn+i,Nm+\ell} = [\mathbf{T}_3]_{Nn+m,Ni+\ell}$. Therefore the elements of $\mathbf{T}_4$ approach zero as M goes to infinity.

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